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Mechanicam vero duplicem Veteres constituerunt: Rationalem *quae per Demonstrationes accurate procedit, & Practicam. Ad practicam spectant Artes omnes Manuales, a quibus utique* Mechanica *nomen mutuata est. Cum autem Artifices parum accurate operari soleant, fit ut* Mechanica *omnis a* Geometria *ita distinguatur, ut quicquid accuratum sit ad* Geometriam *referatur, quicquid minus accuratum ad* Mechanicam. *Attamen errores non sunt Artis sed Artificum. Qui minus accurate operatur, imperfectior est Mechanicus, & si quis accuratissime operari posset, hic foret Mechanicus omnium perfectissimus.* NEWTON

La généralité que j'embrasse, au lieu d'éblouïr nos lumieres, découvrira plutôt les véritables loix de la Nature dans tout leur éclat, et on y trouvera des raisons encore plus fortes, d'en admirer la beauté et la simplicité. EULER

Ceux qui aiment l'Analyse verront avec plaisir la Mechanique en devenir une nouvelle branche ... LAGRANGE

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World Invariant Kinematics

R. A. Toupin

Communicated by C. Truesdell

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1. Introduction

In classical continuum mechanics, the motion of a material medium relative to a rectangular Cartesian system of spatial coordinates z^i, $i=1, 2, 3$, is represented by a set of three single-valued functions $\overset{K}{Z}(z^i, T)$, $K=1, 2, 3$, of the coordinates z^i and the time T. The functions $\overset{K}{Z}$ representing a real continuous motion are subject to the condition

$$\mathfrak{D} \equiv \det \partial_i \overset{K}{Z} \neq 0, \qquad \partial_i \equiv \frac{\partial}{\partial z^i}. \tag{1.1}$$

Every set of three values $Z^K = \overset{K}{Z}(z^i, T)$ for the functions $\overset{K}{Z}$ serves as names for the *material points* $\boldsymbol{P}$ of the continuous medium*. Every set of values for the four *space-time coordinates* (z^i, T) serves as names for an *event* $\boldsymbol{E}$.

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* We shall use the convention that a set of functions like $\overset{K}{Z}(z^i, T)$ will be written with the labeling index set over the kernel letter. When we wish to regard the values of such a set of functions as coordinates we write Z^K. Thus, for example, the partial derivatives of the functions $\overset{K}{Z}$ are again functions $\partial_i \overset{K}{Z}(z^i, T)$ of the coordinates z^i, T. However, from the condition (1.1), it follows that we can solve for the z^i as functions of the Z^K and T. We write the solution in the form $z^i = \overset{i}{z}(Z^K, T)$. The gradients $\partial_i Z^K$ can be regarded as functions of the Z^K and T, in which case we denote these functions by $\partial_i Z^K(Z^K, T) \equiv \partial_i \overset{K}{Z}(\overset{i}{z}(Z^K, T), T)$. This convention avoids excessive use of new symbols and gives a precise meaning to the two different quantities $\partial_i \overset{K}{Z}$ and $\partial_i Z^K$.

The functions $\overset{K}{Z}$ representing the motion specify the material point Z^K which experiences the event (z^i, T). We restrict attention in what follows to local properties of motions. We assume that the functions $\overset{K}{Z}$ are *analytic* at an event $\underset{0}{\boldsymbol{E}}$ with coordinates $\left(\underset{0}{z^i}, \underset{0}{T}\right)$. Thus we assume the existence of a power series

$$\overset{K}{Z}(z^i, T) = \overset{K}{Z}\left(\underset{0}{z^i}, \underset{0}{T}\right) + \overset{K}{Z}_i\left(z^i - \underset{0}{z^i}\right) + \overset{K}{Z}_4\left(T - \underset{0}{T}\right) + \tfrac{1}{2}\overset{K}{Z}_{ij}\left(z^i - \underset{0}{z^i}\right)\left(z^j - \underset{0}{z^j}\right) + \cdots, \quad \overset{K}{Z}_i \neq 0, \tag{1.2}$$

convergent at every event $\boldsymbol{E}$ in some neighborhood $N\left(\underset{0}{\boldsymbol{E}}\right)$ of the event $\underset{0}{\boldsymbol{E}}$. In all that follows, by "analytic" we shall mean analytic at the event $\underset{0}{\boldsymbol{E}}$ and all functions of the coordinates that appear will be assumed analytic at $\underset{0}{\boldsymbol{E}}$. From (1.1) follows the existence of a unique inverse

$$z^i = \overset{i}{z}(Z^K, T). \tag{1.3}$$

The functions $\overset{i}{z}$ will be analytic at the point with coordinate values $\underset{0}{Z^K} = \overset{K}{Z}\left(\underset{0}{z^i}, \underset{0}{T}\right)$, $T = \underset{0}{T}$.

Let $A^i_j(T)$ be any set of nine analytic functions of T such that

$$\delta^{kl} A^i_k A^j_l = \delta^{ij}. \tag{1.4}$$

$A^i_j(T)$ is an orthogonal matrix. Let $d^i(T)$ be any three analytic functions of the time T. Then in terms of a given motion $\overset{K}{Z}(z^i, T)$ we define the *class of all motions* $\overset{K}{Z}{}'(z^i, T)$ *which differ from* $\overset{K}{Z}(z^i, T)$ *by a rigid motion* as follows:

$$\overset{K}{Z}{}'(z^i, T) = \overset{K}{Z}(A^i_j z^j + d^i,\ T + \text{constant}). \tag{1.5}$$

Conversely, any two motions $\left(\overset{K}{Z}{}', \overset{K}{Z}\right)$ between which there exists a relation of the form (1.5) are said to differ from each other by a rigid motion.

An important problem in classical continuum mechanics is the construction of constitutive equations for the stress, internal energy, and heat flux. In general, these constitutive equations depend on a motion and are required to transform in a definite way when a motion is replaced by any motion differing from it by a rigid motion. For example, the theory of finite deformations of elastic media is based on constitutive equations for the stress tensor t^{ij} having the general form [*12*, *8*, *11*]

$$t^{ij}(z^k, T) = \overset{ij}{T}\left(\overset{K}{Z}(z^k, T),\ \partial_m \overset{M}{Z}(z^k, T)\right), \tag{1.6}$$

where, for definiteness, we may assume that the functions $\overset{ij}{T}$ are analytic and single valued in all 12 arguments $\overset{K}{Z}$ and $\partial_i \overset{K}{Z}$. The functional form of $\overset{ij}{T}$ depends on the elastic properties of the material medium. However, for *all* elastic materials the $\overset{ij}{T}$ are required to satisfy the invariance condition

$$\overset{ij}{T}\left(\overset{K}{Z}{}', \partial_m \overset{M}{Z}{}'\right) = A^i_k(T)\, A^j_l(T)\, \overset{kl}{T}\left(\overset{K}{Z}, \partial_m \overset{M}{Z}\right), \tag{1.7}$$

where $(\overset{K}{Z}{}', \overset{K}{Z})$ is any pair of motions differing from each other by a rigid motion. Using known methods of invariant theory one can prove that any set of functions $\overset{ij}{T}$ satisfying (1.7) is reducible to the form

$$\overset{ij}{T} = \frac{\partial \overset{i}{z}}{\partial Z^K} \frac{\partial \overset{j}{z}}{\partial Z^L} \overset{KL}{P}(\overset{M}{Z}, \overset{PQ}{C}, \mathfrak{T}), \qquad \overset{KL}{C} \equiv \delta^{ij} \partial_i \overset{K}{Z} \partial_j \overset{L}{Z} . \tag{1.8}$$

In all that precedes, we have regarded a motion as being relative to a fixed rectangular Cartesian system of spatial coordinates z^i. However, the invariant-theoretic problems of continuum mechanics involving the class of all motions differing from each other by a rigid motion can be attacked from another point of view which is more convenient for our present purposes. Consider the group of coordinate transformations

$$\begin{aligned} z^{i'} &= A^{i'}_j(T)\, z^j + d^{i'}(T), \\ T' &= T + \text{constant}, \end{aligned} \tag{1.9}$$

where $A^{i'}_j(T)$ is an orthogonal matrix. Let $\overset{K}{Z}(z^i, T)$ represent a motion relative to the frame (z^i, T). Let $\overset{K}{Z}{}'(z^{i'}, T')$ be the scalar transform of $\overset{K}{Z}(z^i, T)$. Thus

$$\overset{K}{Z}{}'(z^{i'}, T') = \overset{K}{Z}(z^i, T) \tag{1.10}$$

according to the definition of a scalar in tensor calculus. The functions $\overset{K}{Z}{}'(z^{i'}, T')$ define a motion relative to the frame $(z^{i'}, T')$. The transformation law (1.7) implies that

$$t^{i'j'}(z^{m'}, T') = A^{i'}_k(T)\, A^{j'}_l(T)\, t^{kl}(z^m, T) . \tag{1.11}$$

If we define a set of 16 quantites $\tau^{\mu\nu}$, $\mu, \nu = 1, 2, 3, 4$ by setting

$$\tau^{\mu\nu} = \begin{bmatrix} t^{ij} & 0 \\ 0 & 0 \end{bmatrix}, \tag{1.12}$$

we see that the law of transformation (1.11) for t^{ij} is precisely the law of transformation implied by transforming the quantities $\tau^{\mu\nu}$ as a 4-dimensional tensor under the group of coordinate transformations (1.9) in four variables, where the time T is regarded as the 4$^{\text{th}}$ coordinate. Thus, transforming the functions $\overset{K}{Z}$ representing a motion as a set of three scalars under the group (1.9), we see that the fundamental assumption (1.11) of continuum mechanics assumes a simple and familiar form in terms of a 4-dimensional tensor law of transformation under the group (1.9).

In presenting these familiar ideas we have attempted to indicate the important role in mechanics of the theory of invariants of a motion under a group of coordinate transformations in a 4-dimensional space. In this work we shall introduce three such groups of transformations: the Euclidean group, the Galilean group, and the Lorentzian group. Each of these groups of coordinate transformations on 4 real variables defines a type of geometric space. In each of these spaces, we shall define a motion of a continuous medium. We then define Euclidean kinematics, Galilean kinematics, and Lorentzian kinematics as the

theory of the invariants of a motion in these 4-dimensional spaces. We have attempted to design a formalism which treats these three kinematical theories on a parallel basis. This has been done so that a comparison of classical kinematical concepts, definitions, and theorems with their relativistic counterparts is made easier. Another advantage of this symmetrical treatment is that we can borrow ideas from the more familiar classical invariant theory of a motion and transfer them by analogy to the relativistic case. The Euclidean and Galilean groups are firmly interlocked with classical mechanics; the Lorentzian group, with electromagnetic theory and relativistic mechanics. However, kinematics being the science of motion in itself, independent of the natural laws presumed to govern the motion, we have considered here only briefly the application of kinematical results to these more restricted theories.

2. Euclidean, Galilean, and Lorentzian space-time

So as to fix the meaning we attach to the words *space, geometry, field,* and *invariant,* allow us to describe briefly how they shall be used.

An n-dimensional geometric space $\mathscr{S}$ is a set of points $\boldsymbol{p}$ such that to every point $\boldsymbol{p}$ there corresponds a subset of points $\mathfrak{N}(\boldsymbol{p})$ containing $\boldsymbol{p}$ which can be placed into one to one correspondence with all the ordered sets of n real numbers $x^\mu = (x^1, x^2, \ldots, x^n)$ lying in some interval $\underset{0}{x}^\mu - h^\mu \leqq x^\mu \leqq \underset{0}{x}^\mu + h^\mu$, $h^\mu > 0$ and such that $\boldsymbol{p}$ corresponds to $\underset{0}{x}^\mu$, *together with a group* $\mathfrak{G}$ *of allowable coordinate transformations* $x^{\mu'} = \overset{\mu'}{x}(x^\mu)$, $x^\mu = \overset{\mu}{x}(x^{\mu'})$. The class of all coordinate systems related by elements of the group $\mathfrak{G}$ is called the class of *allowable coordinate systems* for the points $\mathfrak{N}(\boldsymbol{p})$ in the neighborhood of $\boldsymbol{p}$. The characterization of a space $\mathscr{S}$ may involve also a set of functions $\Phi_1, \Phi_2, \ldots, \Phi_N$ of the coordinates x^μ having an assigned transformation law under the group $\mathfrak{G}$. By choosing different groups $\mathfrak{G}$ and different sets of functions $\boldsymbol{\Phi}$ and different transformation laws for the set $\boldsymbol{\Phi}$, we obtain various examples of geometric spaces. Thus ordinary 3-dimensional Euclidean metric space corresponds to letting $\mathfrak{G}$ be the group of orthogonal transformations, where with this choice of $\mathfrak{G}$, the set of functions $\boldsymbol{\Phi}$ is empty. However, we can also let $\mathfrak{G}$ be the group of general analytic transformations $\mathfrak{G}_A$ provided we append a Euclidean metric tensor field $g_{ij}(x^k)$. The two spaces so defined are regarded as equivalent. Curved Riemannian spaces, affinely connected spaces, conformal spaces, *etc.*, correspond to various other choices for the group $\mathfrak{G}$ and the functions $\boldsymbol{\Phi}$ together with their transformation law [*3, 10*]. The foregoing example of Euclidean metric space shows that different choices of the pair of objects $(\mathfrak{G}, \boldsymbol{\Phi})$ may serve to define spaces which are regarded as equivalent.

By a *field* $\boldsymbol{\Phi}$ in a space $\mathscr{S}$ with group $\mathfrak{G}$ we shall mean a set of functions of the coordinates $\Phi_\Omega(x^\mu)$, $\Omega = 1, 2, \ldots, N$ having an assigned law of transformation under the group $\mathfrak{G}$. By "an assigned law of transformation" we mean a rule such that when the representation of the field $\boldsymbol{\Phi}$ by functions $\Phi_\Omega(x^\mu)$ in any one allowable coordinate system x^μ is given, the representation $\Phi_{\Omega'}(x^{\mu'})$ of the field in any other allowable coordinate system is uniquely determined by the functions $\Phi_\Omega(x^\mu)$ and the coordinate transformation relating the $x^{\mu'}$ and x^μ.

A *tensor field* in a space $\mathscr{S}$ with group $\mathfrak{G}$ is a set of functions of the coordinates having a transformation law under $\mathfrak{G}$ of the general form

$$\begin{aligned} \Phi^{\mu'\nu'\cdots}_{\lambda'\cdots\cdots}(x^{\sigma'}) &= |(x'x)|^{-w}\left(\operatorname{sgn}(x'x)\right)^{y}\frac{\partial x^{\mu'}}{\partial x^{\mu}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\cdots\frac{\partial \overset{\lambda}{x}}{\partial x^{\lambda'}}\cdots\Phi^{\mu\nu\cdots}_{\lambda\cdots\cdots}(\overset{\sigma}{x}), \\ (x'x) &\equiv \det\frac{\partial x^{\mu'}}{\partial x^{\nu}}. \end{aligned} \tag{2.1}$$

If $y=0$, $\mathbf{\Phi}$ is called a *tensor field of weight* w. If $y=1$, $\mathbf{\Phi}$ is called an *axial tensor field of weight* w. If $w=y=0$, $\mathbf{\Phi}$ is called an *absolute tensor* field. The number of superscripts and subscripts on a tensor field determine its *contravariant* and *covariant rank*, respectively. The rank of a tensor field is the sum of its contravariant and covariant ranks. Tensors of rank zero are called *scalars*, and tensors of rank one are called *vectors*.

An *affine connection* is a field $\mathbf{\Gamma}$ having a law of transformation of the form [*10*]

$$\Gamma^{\mu'}_{\lambda'\nu'} = \frac{\partial^2 x^{\mu}}{\partial x^{\lambda'}\,\partial x^{\nu'}}\frac{\partial x^{\mu'}}{\partial x^{\mu}} + \frac{\partial x^{\mu'}}{\partial x^{\mu}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\Gamma^{\mu}_{\lambda\nu}. \tag{2.2}$$

We shall have occasion to consider only *symmetric* affine connections: $\Gamma^{\lambda}_{\mu\nu}=\Gamma^{\lambda}_{\nu\mu}$. If $g_{\mu\nu}$ is any symmetric non-singular absolute tensor field, the *Christoffel symbols* based on $g_{\mu\nu}$ are defined by

$$\left\{{\varrho \atop \mu\nu}\right\}_g \equiv \tfrac{1}{2}\,g^{\varrho\lambda}\left(\partial_\nu g_{\mu\lambda}+\partial_\mu g_{\nu\lambda}-\partial_\lambda g_{\mu\nu}\right), \tag{2.3}$$

where $g^{\varrho\lambda}$ is the inverse of $g_{\mu\nu}$, $g^{\varrho\lambda}g_{\lambda\mu}=\delta^{\varrho}_{\mu}$. Christoffel symbols have the transformation law (2.2) of an affine connection. The *Riemann curvature tensor* based on an affine connection $\mathbf{\Gamma}$ is defined by*

$$R_{\dot{\lambda}\dot{\varrho}\nu}{}^{\mu}(\mathbf{\Gamma}) \equiv 2\,\partial_{[\lambda}\Gamma^{\mu}_{\varrho]\nu} + 2\Gamma^{\mu}_{\tau[\lambda}\Gamma^{\tau}_{\varrho]\nu}. \tag{2.4}$$

If $R_{\dot{\lambda}\dot{\varrho}\nu}{}^{\mu}(\mathbf{\Gamma})=0$, the affine connection $\mathbf{\Gamma}$ is said to be *flat* or *integrable*. We denote the Riemann curvature tensor based on the Christoffel symbols $\left\{{\varrho \atop \mu\nu}\right\}_g$ by $R_{\dot{\lambda}\dot{\varrho}\nu}{}^{\mu}(\boldsymbol{g})$. If $R_{\dot{\lambda}\dot{\varrho}\nu}{}^{\mu}(\boldsymbol{g})=0$, and if $g_{\mu\nu}$ is positive definite, the field $g_{\mu\nu}$ is called a *Euclidean metric tensor*.

The covariant derivative of a tensor field based on an affine connection $\mathbf{\Gamma}$ is the tensor field defined by

$$\underset{\Gamma}{\nabla}_{\mu}\Phi^{\lambda\cdots}_{\nu\cdots} \equiv \partial_\mu \Phi^{\lambda\cdots}_{\nu\cdots} + \Gamma^{\lambda}_{\mu\varrho}\Phi^{\varrho\cdots}_{\nu\cdots}\cdots - \Gamma^{\varrho}_{\mu\nu}\Phi^{\lambda\cdots}_{\varrho\cdots}\cdots - w\,\Gamma^{\varrho}_{\varrho\mu}\Phi^{\lambda\cdots}_{\nu\cdots}. \tag{2.5}$$

If the components of the affine connection are Christoffel symbols based on a tensor $g_{\sigma\pi}$, then we write $\underset{g}{\nabla}_{\mu}$ for the operator of covariant differentiation.

An *invariant property* of a field $\mathbf{\Phi}$ in a spaces $\mathscr{S}$ with group $\mathfrak{G}$ is a property possessed in common by each of its representations $\Phi_{\Omega}(x^{\mu})$. The *invariants* of

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* Square brackets enclosing a set of indices denote the alternating sum over all permutations of the enclosed indices divided by $k!$, where k is the number of such indices. Round brackets denote the sum over all permutations divided by $k!$. Thus, for example, $a_{[ij]}\equiv\frac{1}{2}(a_{ij}-a_{ji})$, and $a_{(ij)}\equiv\frac{1}{2}(a_{ij}+a_{ji})$.

a field consist in all of its invariant properties and in other fields which can be defined in terms of it. The *joint invariants* of a set of fields $\mathbf{\Phi}_1, \mathbf{\Phi}_2, \ldots$ consist in all of the invariant properties of the fields held singly and jointly. *Differential invariants* or *joint differential invariants* of a set of fields are joint invariant algebraic relations between the components of the fields and their partial derivatives of all orders with respect to the coordinates.

The *geometry* of a set of fields in a space $\mathscr{S}$ with group $\mathfrak{G}$ is the theory of the joint invariants of the fields under the group $\mathfrak{G}$.

Consider now the three 4-dimensional geometric spaces $\mathscr{S}_{\mathrm{E}}$, $\mathscr{S}_{\mathrm{G}}$, and $\mathscr{S}_{\mathrm{L}}$ defined by the following groups of allowable coordinate transformations on four real variables z^μ, $\mu = 1, 2, 3, 4$*.

I. *Euclidean space-time $\mathscr{S}_{\mathrm{E}}$ and the group $\mathfrak{G}_{\mathrm{E}}$ of Euclidean transformations:*

$$\begin{aligned} z^{i'} &= A^{i'}_j(z^4)\, z^j + d^{i'}(z^4), \\ z^{4'} &= z^4 + \text{constant}, \end{aligned} \tag{2.6}$$

where $A^{i'}_j$ and $d^{i'}$ are analytic functions of z^4 and $A^{i'}_j$ is an orthogonal matrix.

II. *Galilean space-time $\mathscr{S}_{\mathrm{G}}$ and the group $\mathfrak{G}_{\mathrm{G}}$ of Galilean transformations:*

$$\begin{aligned} z^{i'} &= A^{i'}_j\, z^j + u^{i'}\, z^4 + \text{constant}, \\ z^{4'} &= z^4 + \text{constant}, \end{aligned} \tag{2.7}$$

where $A^{i'}$ is a constant orthogonal matrix and the $u^{i'}$ are constants.

III. *Lorentzian space-time $\mathscr{S}_{\mathrm{L}}$ and the group $\mathfrak{G}_{\mathrm{L}}$ of Lorentz transformations:*

$$z^{\mu'} = L^{\mu'}_\nu\, z^\nu, \tag{2.8}$$

where $L^{\mu'}_\nu$ is a constant matrix satisfying the equations

$$\eta^{\mu\nu}\, L^{\lambda'}_\mu\, L^{\tau'}_\nu = \eta^{\lambda'\tau'}, \tag{2.9}$$

$$\eta^{\mu\nu} = \eta^{\mu'\nu'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

A *motion of a material medium* $\boldsymbol{M}$ *in Euclidean or Galilean space-time* is defined by any three absolute scalar fields $\overset{K}{Z}(z^\mu)$, $K = 1, 2, 3$ such that the matrix $\overset{KL}{C^{-1}}(z^\mu)$ defined by

$$\overset{KL}{C^{-1}} \equiv \delta^{ij}\, \partial_i \overset{K}{Z}\, \partial_j \overset{L}{Z} \tag{2.10}$$

is *positive definite.*

* From this point on, Greek lower case indices will always range over the four values 1, 2, 3, 4. Greek upper case indices will be reserved for a variable range depending on the context. Latin lower and upper case indices will always range over the three values 1, 2, 3. The summation convention applies to all types of indices.

A motion of a material medium $\mathbf{M}$ *in Lorentzian space-time* is defined by any three absolute scalar fields $\overset{K}{Z}(z^\mu)$ such that the matrix $\overset{KL}{X^{-1}}(z^\mu)$ defined by

$$\overset{KL}{X^{-1}} \equiv \eta^{\mu\nu}\,\partial_\mu \overset{K}{Z}\,\partial_\nu \overset{L}{Z} \tag{2.11}$$

is *positive definite, i.e.,* $\overset{KL}{X^{-1}} V_K V_L > 0$ for all $V_K \neq 0$*.

The geometry of a motion in $\mathscr{S}_\mathrm{E}$, $\mathscr{S}_\mathrm{G}$, and $\mathscr{S}_\mathrm{L}$ will be called *Euclidean kinematics, Galilean kinematics,* and *Lorentzian kinematics,* respectively.

3. Klein's principle and general coordinates in Euclidean, Galilean, and Lorentzian space-time

The use of curvilinear coordinates for 3-dimensional Euclidean space is familiar in mechanics. Many authors in continuum mechanics, especially in finite elasticity theory [*11*], use curvilinear and *deforming* spatial coordinates x^i in Galilean space-time. This type of coordinate system is related to an inertial rectangular Cartesian coordinate system z^i by a general analytic transformation of the form

$$\begin{aligned} x^i &= \overset{i}{x}(z^i, z^4), \\ x^{4'} &= z^4 + \text{constant}. \end{aligned} \tag{3.1}$$

Unless the transformation (3.1) is a Galilean transformation, the spatial coordinate system x^i is said to be non-inertial or curvilinear, or both non-inertial and curvilinear. Non-inertial spatial coordinate systems in classical mechanics are further classified as rigid, deforming, accelerated, rotating, *etc.* Though the use of non-inertial curvilinear spatial coordinates is accepted practice in classical mechanics, the fourth coordinate (time) is rarely transformed more generally than in a Galilean transformation. Thus there has arisen a large body of literature [*5, 7*] concerned with the invariants of a motion under the more general group of transformations (3.1). Now the Euclidean and Galilean groups are subgroups of the more general transformations (3.1); however, the Lorentz group is not a subgroup of (3.1) since the fourth coordinate in a Lorentz transformation is transformed more generally than in $(3.1)_2$. The utility of introducing a more general class of coordinates than the z^μ in space-time once accepted, there seems little motivation for giving undue special attention to the group (3.1) in this work, which attempts a uniform treatment of Euclidean, Galilean, and Lorentzian kinematics. What we shall do is to develop a formalism for kinematics in terms of invariants under the *group* $\mathfrak{G}_\mathrm{A}$ *of unrestricted analytic transformations on all four coordinates of events.* General coordinates in space-time will be denoted by x^μ and a typical element of the group $\mathfrak{G}_\mathrm{A}$ is written in the form

$$x^{\mu'} = \overset{\mu}{x}(x^\mu), \qquad x^\mu = \overset{\mu'}{x}(x^{\mu'}). \tag{3.2}$$

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* In § 7 we introduce a group of transformations $Z^{K'} = \overset{K'}{H}(Z^K)$ of the material coordinates Z^K. At the appropriate point in the discussion, it will be shown how a motion in any of these spaces determines an inverse relation $z^\mu = \overset{\mu}{z}(Z^K, \tau)$ between the space-time coordinate z^μ, the material coordinates Z^K and a suitable fourth variable τ. When the scalar fields $\overset{KL}{C^{-1}}$ and $\overset{KL}{X^{-1}}$ are considered as functions of the Z^K and τ, we shall write $(C^{-1})^{KL}$, $(X^{-1})^{KL}$ consistent with the fact that these quantities, so regarded, transform as tensors under transformations of the material coordinates.

The groups $\mathfrak{G}_E$, $\mathfrak{G}_G$, and $\mathfrak{G}_L$ are all subgroups of $\mathfrak{G}_A$. Once such a formalism for the three kinematical systems has been developed, the classical problem of introducing moving and deforming coordinates is of course solved since (3.1) is a subgroup of (3.2). That is, any invariant of a motion under (3.2) is automatically an invariant under the subgroup (3.1) of these more general transformations. The concepts needed to construct such a formalism for kinematics are embodied in Klein's *principle* [*10*, p. 65]:

If in any space with group $\mathfrak{G}_1$ the subgroup $\mathfrak{G}_2$ is introduced, consisting in all transformations which leave a figure (field) $\mathbf{\Phi}_1$ invariant, then the geometry of a figure $\mathbf{\Phi}_2$ with respect to $\mathfrak{G}_2$ is identical with the geometry of the set of figures $(\mathbf{\Phi}_1, \mathbf{\Phi}_2)$ with respect to $\mathfrak{G}_1$.

Let us illustrate the application of Klein's principle that we intend to make by the following familiar example. Suppose we have given a tensor field $f^{i\cdots}_{j\cdots}$ in ordinary 3-dimensional Euclidean metric space where the group $\mathfrak{G}_2$ is the orthogonal group, *i.e.*, $f^{i\cdots}_{j\cdots}$ is a Cartesian tensor. Let $\mathfrak{G}_1$ be the group of general analytic coordinate transformations in 3-dimensions. $\mathfrak{G}_2$ is a subgroup of $\mathfrak{G}_1$. Let $g_{ij}(x^k)$ be an absolute symmetric positive definite tensor field under $\mathfrak{G}_1$ such that its Riemann curvature tensor vanishes. Then in the space with group $\mathfrak{G}_1$ we know [*6*, § 10] that there exist preferred coordinate systems z^i such that $g_{ij}(z^k) = \delta_{ij}$. Furthermore, any such pair of coordinate systems are related to each other by an orthogonal transformation. Thus the group $\mathfrak{G}_2$ can be defined as the subgroup of $\mathfrak{G}_1$ which leaves the *canonical form* δ_{ij} of the Euclidean metric field g_{ij} invariant. Let $\varphi^{i\cdots}_{j\cdots}(x^k)$ be any field in the space with group $\mathfrak{G}_1$ having any law of transformation under $\mathfrak{G}_1$ such that

$$\varphi^{i\cdots}_{j\cdots}(z^k) = f^{i\cdots}_{j\cdots}(z^k) \tag{3.3}$$

in every preferred coordinate system in which $g_{ij} = \delta_{ij}$. According to Klein's principle, the theory of the invariants of the field $f^{i\cdots}_{j\cdots}$ under the group $\mathfrak{G}_2$ is identical with the theory of the joint invariants of the fields $(\varphi^{i\cdots}_{j\cdots}, g_{km})$ under the group $\mathfrak{G}_1$ of general analytic transformations or under any group containing $\mathfrak{G}_2$ as a subgroup.

Consider the group $\mathfrak{G}_A$ of general analytic transformations (3.2) of the four coordinates of events. Our objective is to define three sets of fields $\{\mathbf{\Phi}\}_\Delta$, $\Delta = \mathrm{E}$, G, L, having an assigned law of transformation under $\mathfrak{G}_A$ such that (1) there exists a subclass of preferred coordinates z^μ in which the fields $\{\mathbf{\Phi}\}_\Delta$ assume certain canonical forms and (2) the subgroups $\mathfrak{G}_E$, $\mathfrak{G}_G$, and $\mathfrak{G}_L$ consist in all the transformations of $\mathfrak{G}_A$ which leave invariant the canonical forms of the sets of fields $\{\mathbf{\Phi}\}_E$, $\{\mathbf{\Phi}\}_G$, and $\{\mathbf{\Phi}\}_L$, respectively. Once we determine such a set of fields we invoke Klein's principle and give new but equivalent definitions of Euclidean, Galilean, and Lorentzian kinematics. That is, these three theories can then be defined as the theories of the joint invariants of the combined sets of fields $\overset{K}{Z}(x^\mu)$, $\{\mathbf{\Phi}\}_\Delta(x^\mu)$ under the group $\mathfrak{G}_A$.

Case I. Euclidean space-time. Let $t_\mu(x^\nu) \not\equiv 0$ be an absolute covariant vector field under $\mathfrak{G}_A$ such that

$$\partial_{[\mu} t_{\nu]} = 0. \tag{3.4}$$

Let $g^{\mu\nu}(x^\pi)$ be a symmetric contravariant singular absolute tensor field under $\mathfrak{G}_A$ such that

$$g^{\mu\nu} t_\nu = 0, \qquad g^{\mu\nu} v_\mu v_\nu > 0, \tag{3.5}$$

for all $v_\mu \neq 0$ and not parallel to t_ν. The condition (3.4) is necessary and sufficient for the existence of a scalar field $t(x^\mu)$ such that

$$t_\mu = \partial_\mu t. \tag{3.6}$$

Moreover, the field $t(x^\mu)$ is uniquely determined by (3.6) to within an additive constant. Let $\overset{i}{y}(x^\mu)$, $i = 1, 2, 3$ be any three analytic functions of the coordinates such that*

$$\boldsymbol{\Theta} = \varepsilon^{\mu\nu\varrho\tau} \partial_\mu \overset{1}{y} \, \partial_\nu \overset{2}{y} \, \partial_\varrho \overset{3}{y} \, \partial_\tau t \neq 0. \tag{3.7}$$

From (3.7) it follows that we can solve for the coordinates x^μ as functions of the variables y^i and $T = t(x^\mu)$

$$x^\mu = \overset{\mu}{x}(y^i, T). \tag{3.8}$$

Consider the functions $g^{ij}(y^k, T)$ defined by

$$g^{ij}(y^k, T) \equiv g^{\mu\nu}(\overset{\pi}{x}) \, \partial_\mu y^i \, \partial_\nu y^j. \tag{3.9}$$

We assume that the Riemann curvature tensor based on the positive definite symmetric $g^{ij}(y^k, T)$ vanishes for all values of y^i and T corresponding to the events in $N(\underset{0}{\boldsymbol{E}})$,

$$R_{jkl}{}^{i}(\boldsymbol{g}) = 0. \tag{3.10}$$

These conditions are necessary and sufficient that we be able to choose functions $y^{i'} = \overset{i'}{y}(y^i, T)$ such that the $g^{i'j'}$ defined with respect to the $y^{i'}$ have values $g^{i'j'} = \delta^{i'j'}$. Thus there exists a coordinate transformation

$$\begin{aligned} z^{i'} &= \overset{i'}{z}(x^\mu), \\ z^{4'} &= t(x^\mu) + \text{constant}, \end{aligned} \tag{3.11}$$

such that in the coordinate system $z^{\mu'}$ the fields $g^{\mu'\nu'}(z^{\pi'})$ and $t_{\mu'}(z^{\pi'})$ have the *canonical form*

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad t_\mu = (0, 0, 0, 1). \tag{3.12}$$

Applying the assumed tensor law of transformation to these canonical forms we then see that the Euclidean group of transformations (2.6) can be defined as the subgroup of $\mathfrak{G}_A$ which leaves these canonical forms invariant.

Thus *Euclidean kinematics is the theory of the joint invariants under the group* $\mathfrak{G}_A$ *of the set of fields*

$$\overset{K}{Z}(x^\mu), \qquad g^{\mu\nu}(x^\pi), \qquad t_\mu(x^\pi), \tag{3.13}$$

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* $\varepsilon^{\mu\nu\varrho\tau}$ and $\varepsilon_{\mu\nu\varrho\tau}$ are the completely antisymmetric axial tensor fields of weights $+1$ and -1, respectively, whose components are $+1$, -1, or 0 in every coordinate system and $\varepsilon^{1234} = \varepsilon_{1234} = +1$.

where the condition (2.10) invariant under $\mathfrak{G}_E$ is replaced by the condition

$$\overset{KL}{C^{-1}} V_K V_L > 0, \quad V_K \neq 0, \quad \overset{KL}{C^{-1}} \equiv g^{\mu\nu} \partial_\mu \overset{K}{Z} \partial_\nu \overset{L}{Z}. \tag{3.14}$$

A coordinate system z^μ in Euclidean space-time such that (3.12) holds will be called a *Euclidean frame*. We shall call $g^{\mu\nu}$ the *space metric*, and we shall call t_μ the *covariant space normal*.

Case II. Galilean space-time. Let $\Gamma^\lambda_{\mu\nu}(x^\pi)$ be an affine connection under $\mathfrak{G}_A$. We assume that $\mathbf{\Gamma}$ is a flat or integrable connection.

$$R_{\lambda\varrho\nu}{}^{\mu}(\mathbf{\Gamma}) = 0. \tag{3.15}$$

Let $g^{\mu\nu}$ and t_μ be tensors under $\mathfrak{G}_A$ having the same properties assigned to these fields as in Case I above, but which in addition satisfy the conditions

$$\begin{aligned} \underset{\Gamma}{\nabla_\lambda} g^{\mu\nu} &= \partial_\lambda g^{\mu\nu} + \Gamma^\mu_{\lambda\varrho} g^{\varrho\nu} + \Gamma^\nu_{\lambda\varrho} g^{\mu\varrho} = 0, \\ \underset{\Gamma}{\nabla_\nu} t_\mu &= \partial_\nu t_\mu - \Gamma^\lambda_{\mu\nu} t_\lambda = 0, \end{aligned} \tag{3.16}$$

jointly with the connection $\Gamma^\lambda_{\mu\nu}$. That is, the covariant derivatives of $g^{\mu\nu}$ and t_μ based on the connection $\Gamma^\lambda_{\mu\nu}$ vanish identically.

From (3.15) follows the existence of preferred coordinate systems in which all of the components of the connection vanish [*1*, § 29]. Any two such systems are related by a *linear* transformation. From (3.16) it follows that, in any of the coordinate systems in which the connection vanishes, the components of $g^{\mu\nu}$ and t_μ are constants. Set $z^4 = t(x^\mu)$. This will be a linear transformation leaving the connection zero, and t_μ will assume its canonical form

$$t_\mu = (0, 0, 0, 1). \tag{3.17}$$

From (3.5) it then follows that $g^{\mu\nu}$ is reduced to the form

$$g^{\mu\nu} = \begin{bmatrix} g^{ij} & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.18}$$

where g^{ij} is a constant symmetric positive definite matrix. Thus by a further linear transformation of the first three coordinates not involving z^4, preserving the condition (3.17), and the vanishing of the connection, we can reduce $g^{\mu\nu}$ to its canonical form

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.19}$$

It is then an easy matter to verify that the Galilean group (2.7) is the subgroup of $\mathfrak{G}_A$ which leaves invariant all three canonical forms (3.17), (3.18), and $\Gamma^\lambda_{\mu\nu} = 0$.

Thus *Galilean kinematics is the theory of the joint invariants under* $\mathfrak{G}_A$ *of the set of fields*

$$\overset{K}{Z}(x^\mu), \quad g^{\mu\nu}(x^\pi), \quad t_\mu(x^\pi), \quad \Gamma^\lambda_{\mu\nu}(x^\pi), \tag{3.20}$$

where the $\overset{K}{Z}$ satisfy the invariant condition (3.14). The preferred coordinate systems z^μ in Galilean space-time in which we have (3.17), (3.19), and $\Gamma^\varrho_{\mu\nu} = 0$ will be called *Galilean frames*, and $\Gamma^\mu_{\nu\lambda}$ will be called the *Galilean connection*.

Case III. Lorentzian space-time. Let $\gamma^{\mu\nu}(x^\pi)$ be a non-singular symmetric tensor field with signature 2 whose Riemann curvature tensor vanishes. These are necessary and sufficient conditions for the existence of preferred coordinates z^μ such that [*6*, § 27]

$$\gamma^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{3.21}$$

The Lorentz transformations (2.8) consists in the group of all transformations of $\mathfrak{G}_A$ which leave the canonical form (3.21) for $\gamma^{\mu\nu}$ invariant.

Thus *Lorentzian kinematics is the theory of the joint invariants under* $\mathfrak{G}_A$ *of the set of fields*

$$\overset{K}{Z}(x^\mu), \quad \gamma^{\mu\nu}(x^\mu), \tag{3.22}$$

where the condition (2.11) invariant under G_L is replaced by the condition

$$\overset{KL}{X^{-1}} V_K V_L > 0, \quad V_K \neq 0, \quad \overset{KL}{X^{-1}} \equiv \gamma^{\mu\nu} \partial_\mu \overset{K}{Z} \partial_\nu \overset{L}{Z}, \tag{3.23}$$

invariant under $\mathfrak{G}_A$. The preferred coordinates z^μ in Lorentzian space-time in which we have (3.21) will be called *Lorentz frames*. The tensor field $\gamma^{\mu\nu}$ and its inverse $\gamma_{\mu\nu}$ will be called the *Lorentz metric*.

Thus we have succeeded in formulating all three kinematical systems as theories of the joint invariants of a motion and a suitable set of fields under a common group of coordinate transformations $\mathfrak{G}_A$. Quantities transforming as a tensor under $\mathfrak{G}_A$ will be called *world tensors*. An affine connection under $\mathfrak{G}_A$ will be called a *world connection*. The Galilean connection is a world connection, $g^{\mu\nu}$, t_μ, and $\gamma^{\mu\nu}$ are world tensors, and the $\overset{K}{Z}$ are world scalars.

4. Euclidean and Galilean kinematics

The results we present in this section and the following one are not intended to represent an exhaustive, systematic study of the invariants of a motion in $\mathscr{S}_E$, $\mathscr{S}_G$, and $\mathscr{S}_L$. Rather, the remarks and equations in these sections are intended merely to illustrate the world invariant formalism and to show the ease with which familiar results, often obtained otherwise by cumbersome methods, follow easily and elegantly.

Since the Galilean group is a subgroup of the Euclidean group and a motion is defined in just the same way in both spaces, it is clear that any Euclidean invariant of a motion is also a Galilean invariant. Comparing the lists of fields (3.13) and (3.20) we see that, in the world invariant formalism, this means that any joint invariant of the fields (3.13) is also a joint invariant of the fields (3.20). Thus it is appropriate that we discuss these two kinematical theories simultaneously. Any invariant of a motion in $\mathscr{S}_E$ or $\mathscr{S}_G$ which does not depend on the Galilean connection is a Euclidean invariant of the motion. However, if an invariant depends explicitly on the Galilean connection, then this invariant will be a Galilean invariant of the motion but not a Euclidean invariant of the motion.

Euclidean kinematics as we have defined it here is equivalent to an invariant theory proposed by DEFRISE [*20*]. We have announced the problem somewhat differently, but the two theories are in fact the same. The paper by DEFRISE contains a number of geometrical results pertaining to a motion in Euclidean space-time. Some of these will be included in the discussion here. The references contain other sources of related material.

Consider the axial world scalar of weight 1 defined by

$$\mathfrak{D} \equiv \frac{1}{3!}\, \varepsilon^{\mu\nu\varrho\tau}\, \partial_\mu \overset{K}{Z}\, \partial_\nu \overset{L}{Z}\, \partial_\varrho \overset{M}{Z}\, \partial_\tau t\, \varepsilon_{KLM} \tag{4.1}$$

and the axial world vector of weight 1 defined by

$$\mathfrak{v}^\mu = \frac{1}{3!}\, \varepsilon^{\nu\varrho\tau\mu}\, \partial_\nu \overset{K}{Z}\, \partial_\varrho \overset{L}{Z}\, \partial_\tau \overset{M}{Z}\, \varepsilon_{KLM}. \tag{4.2}$$

In a Euclidean frame, $\mathfrak{D} = \det \partial_i \overset{K}{Z} = \pm \sqrt{\det \overset{KL}{C^{-1}}} \neq 0$. Since the law of transformation for $\mathfrak{D}$ is $\mathfrak{D}' = (x'\,x)^{-1}\mathfrak{D}$ and $(x'\,x)$ is never zero, $\mathfrak{D} \neq 0$ in any coordinate system. Thus we can define the absolute world contravariant vector field

$$v^\mu \equiv \frac{\mathfrak{v}^\mu}{\mathfrak{D}} \tag{4.3}$$

called the *world velocity vector of the motion*. The form which any world tensor or other type of world invariant takes in every Euclidean, Galilean, or Lorentz frame, depending on the context, will be called its *canonical form*. The canonical form of the world velocity vector v^μ is

$$v^\mu = (v^i, 1). \tag{4.4}$$

Since $\mathfrak{D}$ is never zero, we can always solve for any system of general coordinates x^μ in terms of the material coordinates $Z^K = \overset{K}{Z}$ and the *time* $T = t(x^\mu)$. Thus we always have relations of the form

$$x^\mu = \overset{\mu}{x}(Z^K, T), \tag{4.5}$$

where the functions $\overset{\mu}{x}$ are single-valued and analytic. In terms of the $\overset{\mu}{x}$ the world velocity vector v^μ is given by

$$v^\mu = \frac{\partial x^\mu}{\partial T} = \overset{\mu}{v}(\overset{K}{Z}, t), \qquad \overset{\mu}{v}(Z^K, T) \equiv \frac{\partial \overset{\mu}{x}}{\partial T}. \tag{4.6}$$

The result (4.5) serves to promote the geometric interpretation of a motion in terms of a congruence of lines in space-time which are nowhere tangent to the surfaces $t(x^\mu) = T = \text{constant}$. Such a surface is called *an instantaneous space*. The material coordinates Z^K serve as names for the lines of the congruence and T is an admissible parameter whose value is never stationary as one moves along a line of the congruence. A line of the congruence (4.5) is called the *world line* of the corresponding material point Z^K. Each surface $t(x^\mu) = T$ is an ordinary 3-dimensional Euclidean metric space imbedded in 4-dimensional Euclidean or Galilean space-time. One can introduce a general system of parameters or *instantaneous space coordinates*

$$y^i = \overset{i}{y}(x^\mu) \tag{4.7}$$

on each of the one parameter family of surfaces $t(x^\mu) = T$. We can also arrange matters so that the x^μ are given in terms of the y^i and T by functions

$$x^\mu = \overset{\mu}{x}(y^i, T) \tag{4.8}$$

analytic in all four variables y^i and T. The material coordinates Z^K of a material medium constitute one such set of instantaneous space coordinates. The induced surface metric $g^{ij}(y^k, T)$ defined by

$$g^{ij}(y^k, T) \equiv g^{\mu\nu}(\overset{\pi}{x})\, \partial_\mu y^i\, \partial_\nu y^j \tag{4.9}$$

is always Euclidean. Let the functions $(C^{-1})^{KL}(Z^M, T)$ be defined by (*cf.* the remarks in the footnote, page 186)

$$(C^{-1})^{KL}(Z^M, T) \equiv \overset{KL}{C^{-1}}(\overset{\mu}{x}(Z^M, T)), \tag{4.10}$$

where $\overset{KL}{C^{-1}}$ is the set of scalar functions defined in (3.14) and the $\overset{\mu}{x}$ are the functions occurring in (4.5). If we choose material coordinates Z^K for the instantaneous space coordinates y^i, the components of the instantaneous space metric $g^{KL}(Z^M, T)$ are obviously given by

$$g^{KL}(Z^M, T) = (C^{-1})^{KL}(Z^M, T). \tag{4.11}$$

Thus for any motion in Euclidean space-time or Galilean space-time we have

$$R_{\dot K \dot L \dot M}{}^N(C_{PQ}) = 0. \tag{4.12}$$

The quantities $(C^{-1})^{KL}$ or the inverse C_{KL}, $C_{KL}(C^{-1})^{LM} = \delta^M_K$, are called *material measures of deformation* [8, p. 140].

A motion in Euclidean or Galilean space-time is called *rigid* if and only if the functions $(C^{-1})^{KL}$ are independent of the time T, *i.e.*,

$$(\dot{C^{-1}})^{KL} = \frac{\partial (C^{-1})^{KL}}{\partial T} = v^\mu\, \partial_\mu \overset{KL}{C^{-1}} = 0. \tag{4.13}$$

Let $g_{KL}(Z^M, T)$ denote the inverse of g^{KL}. The distance between two neighboring material points Z^K and $Z^K + dZ^K$ at time T is given by

$$dS^2 = g_{KL}\, dZ^K\, dZ^L = C_{KL}\, dZ^K\, dZ^L. \tag{4.14}$$

Thus a motion is rigid if and only if $d\dot S = 0$ for every pair of neighboring material points.

The *Lie derivative* [10, p. 106] of a tensor field $\varphi^{\mu\cdots}_{\nu\cdots}$ with respect to an absolute contravariant vector field v^μ is a tensor field of the same type as $\varphi^{\mu\cdots}_{\nu\cdots}$ defined by

$$\underset{v}{\pounds}\, \varphi^{\mu\cdots}_{\nu\cdots} \equiv v^\lambda\, \partial_\lambda \varphi^{\mu\cdots}_{\nu\cdots} - \varphi^{\lambda\cdots}_{\nu\cdots}\, \partial_\lambda v^\mu - \cdots + \varphi^{\mu\cdots}_{\lambda\cdots}\, \partial_\nu v^\lambda + \cdots + w\, \partial_\lambda v^\lambda\, \varphi^{\mu\cdots}_{\nu\cdots}. \tag{4.15}$$

Consider then the contravariant absolute symmetric world tensor $\Delta^{\mu\nu}$ defined by

$$\begin{aligned} \Delta^{\mu\nu} &\equiv -\tfrac{1}{2}\, \underset{v}{\pounds}\, g^{\mu\nu} \\ &\equiv -\tfrac{1}{2}\,(v^\lambda\, \partial_\lambda g^{\mu\nu} - g^{\lambda\nu}\, \partial_\lambda v^\mu - g^{\mu\lambda}\, \partial_\lambda v^\nu), \end{aligned} \tag{4.16}$$

where v^μ is the world velocity vector of the motion. The tensor $\Delta^{\mu\nu}$ has the canonical form

$$\Delta^{\mu\nu} = \begin{bmatrix} d^{ij} & 0 \\ 0 & 0 \end{bmatrix}, \qquad d^{ij} = \tfrac{1}{2}(\partial_i v^j + \partial_j v^i). \tag{4.17}$$

The quantities d^{ij} are the familiar Cartesian components of the *rate of deformation tensor* [8, p. 150]. Since $\Delta^{\mu\nu}$ has the canonical form (4.17), we shall call it the *world rate of deformation tensor*. Since $\Delta^{\mu\nu}$ is defined independently of the Galilean connection, it is a Euclidean invariant as well as a Galilean invariant of a motion★.

The world scalar invariant equation

$$v^\mu \partial_\mu \overset{KL}{C^{-1}} = -2\Delta^{\mu\nu} \partial_\mu \overset{K}{Z} \partial_\nu \overset{L}{Z} \tag{4.18}$$

can be easily verified by referring all quantities to a Euclidean frame. Thus a sufficient condition for a motion to be rigid is that $\Delta^{\mu\nu}=0$. This condition is also necessary, for on referring all quantities to a Euclidean frame we get

$$v^\mu \partial_\mu \overset{KL}{C^{-1}} = 0 = -2d^{ij} \partial_i \overset{K}{Z} \partial_j \overset{L}{Z}, \qquad \det \partial_i \overset{K}{Z} \neq 0, \tag{4.19}$$

from which it follows that $d^{ij}=0$. Thus every component of $\Delta^{\mu\nu}$ vanishes in a Euclidean frame if the motion is rigid. Since $\Delta^{\mu\nu}$ transforms as a tensor under general transformations of the coordinates, its components will vanish in every system of coordinates if the motion is rigid. Thus *the vanishing of the world rate of deformation tensor $\Delta^{\mu\nu}$ is a necessary and sufficient condition that a motion in $\mathcal{S}_{\mathrm{E}}$ or $\mathcal{S}_{\mathrm{G}}$ be rigid.*

The field $\mathfrak{g}$ defined by

$$\mathfrak{g} \equiv \frac{1}{3!} \varepsilon_{\mu\varrho\lambda\tau} \varepsilon_{\nu\omega\zeta\varphi} g^{\varrho\omega} g^{\lambda\zeta} g^{\tau\varphi} v^\mu v^\nu \tag{4.20}$$

is a world scalar of weight -2 having the constant value $\mathfrak{g}=1$ in every Euclidean frame. The familiar absolute scalar invariants of the rate of deformation tensor d^{ij} are given in world invariant form by the formulae

$$\Theta_1 = \frac{\mathfrak{g}^{-1}}{2} \varepsilon_{\mu\varrho\lambda\tau} \varepsilon_{\nu\omega\zeta\varphi} g^{\varrho\omega} g^{\lambda\zeta} \Delta^{\mu\nu} v^\tau v^\varphi, \tag{4.21}$$

$$\Theta_2 = \frac{\mathfrak{g}^{-1}}{2!} \varepsilon_{\mu\varrho\lambda\tau} \varepsilon_{\nu\omega\zeta\varphi} g^{\varrho\omega} \Delta^{\lambda\zeta} \Delta^{\mu\nu} v^\tau v^\varphi, \tag{4.22}$$

$$\Theta_3 = \frac{\mathfrak{g}^{-1}}{3!} \varepsilon_{\mu\varrho\lambda\tau} \varepsilon_{\nu\omega\zeta\varphi} \Delta^{\varrho\omega} \Delta^{\lambda\zeta} \Delta^{\mu\nu} v^\tau v^\varphi. \tag{4.23}$$

The canonical form of these absolute scalars is

$$\Theta_1 = \operatorname{trace} d^{ij}, \qquad \Theta_2 = \text{sum of the principal minors of } d^{ij}, \qquad \Theta_3 = \det d^{ij}. \tag{4.24}$$

As an illustration of how the theory may be applied to problems in mechanics, consider the world tensor $\tau^{\mu\nu}$ defined by

$$\tau^{\mu\nu} = \lambda \Theta_1 g^{\mu\nu} + 2\mu \Delta^{\mu\nu} - p g^{\mu\nu}. \tag{4.25}$$

The canonical form of this tensor is

$$\tau^{\mu\nu} = \begin{bmatrix} t^{ij} & 0 \\ 0 & 0 \end{bmatrix}, \qquad t^{ij} = \lambda d^{kk} \delta^{ij} + 2\mu d^{ij} - p \delta^{ij}. \tag{4.26}$$

★ In pure geometry (see, *e.g.*, [*10*, p. 346]), a motion in a space characterized by a set of fields $\varphi^{\mu\cdots}_{\nu\cdots}$ is a vector field w^π satisfying one or more equations of the type $\underset{w}{\pounds}\, \varphi^{\mu\cdots}_{\nu\cdots} = 0$. Thus we are using the word motion in quite a different sense here. However, much of the theory of motions in the sense of pure geometry can be applied to the study of motions of continuous media.

The quantities t^{ij} transform according to required law (1.11) under the group $\mathfrak{G}_E$ relating the Euclidean frames. Equations $(4.26)_2$ are the familiar constitutive equations for the stress tensor of a classical linear viscous fluid, where λ and μ are the viscosities and p is the pressure [8, p. 126]. The pressure is assumed to be some function of the world scalars $\overset{KL}{C^{-1}}$ and the temperature. From the point of view of world invariant kinematics, the stress $\tau^{\mu\nu}$ is a world tensor differential invariant of a motion in Euclidean space-time satisfying the invariant equations $\tau^{\mu\nu} = \tau^{\nu\mu}$, $\tau^{\mu\nu} t_\nu = 0$.

The Lie derivative of the world rate of deformation tensor $\Delta^{\mu\nu}$ is again a tensor given by

$$\overset{*}{\Delta}{}^{\mu\nu} \equiv \underset{v}{\pounds}\, \Delta^{\mu\nu} = v^\lambda \partial_\lambda \Delta^{\mu\nu} - \Delta^{\lambda\nu} \partial_\lambda v^\mu - \Delta^{\mu\lambda} \partial_\lambda v^\nu. \tag{4.27}$$

The canonical form of $\overset{*}{\Delta}$ is

$$\overset{*}{\Delta}{}^{\mu\nu} = \begin{bmatrix} \overset{*}{d}{}^{ij} & 0 \\ 0 & 0 \end{bmatrix}, \quad \overset{*}{d}{}^{ij} = \frac{\partial d^{ij}}{\partial z^4} + \partial_l d^{ij} v^l - d^{lj} \partial_l v^i - d^{il} \partial_l v^j. \tag{4.28}$$

The quantities $\overset{*}{d}{}^{ij}$ transform as a 3-dimensional tensor under $\mathfrak{G}_E$. We see that the above process may be continued indefinitely to obtain an infinite sequence of world tensors $\Delta^{\mu\nu}, \overset{*}{\Delta}{}^{\mu\nu}, \ldots, \overset{(n)}{\Delta}{}^{\mu\nu}, \ldots$. All of these fields have a canonical form similar to (4.28) involving a sequence of 3-dimensional tensor fields $d^{ij}, \overset{*}{d}{}^{ij}, \ldots, \overset{(n)}{d}{}^{ij}, \ldots$. Such a sequence of differential invariants of a motion under the group $\mathfrak{G}_E$ has been considered by ERICKSEN & RIVLIN [*12*] in the formulation of constitutive equations for the stress in a visco-elastic material. If $\tau^{\mu\nu}$ is *any* tensor invariant of a motion in Euclidean space-time such that it has the canonical form $(4.26)_1$, then t^{ij} is an admissible stress tensor defined in terms of the motion and satisfying the transformation law (1.11) under the group $\mathfrak{G}_E$. Thus we see how the present formalism can be put to use in the problem of formulating admissible consititutive relations for the stress tensor in classical continuum mechanics. The Lie derivative of $\tau^{\mu\nu}$ with respect to the world velocity v^μ has the canonical form

$$\overset{*}{\tau}{}^{\mu\nu} = \begin{bmatrix} \overset{*}{t}{}^{ij} & 0 \\ 0 & 0 \end{bmatrix}, \quad \overset{*}{t}{}^{ij} = \frac{\partial t^{ij}}{\partial z^4} + \partial_l t^{ij} v^l - t^{lj} \partial_l v^i - t^{il} \partial_l v^j. \tag{4.29}$$

The quantities $\overset{*}{t}{}^{ij}$ transform as a tensor under $\mathfrak{G}_E$ and have been used in formulating the constitutive equations of a class of materials called hypo-elastic [*16, 21*]. The tensor $\overset{*}{t}{}^{ij}$ is called the *convected time flux* of the stress tensor. The proof of its invariance under $\mathfrak{G}_E$ has been discussed from numerous points of view [*7, 8, 13, 14, 21*].

In terms of the world tensors $g^{\mu\nu}$ and v^μ we can define the non-singular symmetric contravariant world tensor $p^{\mu\nu}$ given by

$$p^{\mu\nu} \equiv g^{\mu\nu} - v^\mu v^\nu. \tag{4.30}$$

Let $p_{\mu\nu}$ denote the inverse of $p^{\mu\nu}$. The canonical form of $p_{\mu\nu}$ is

$$p_{\mu\nu} = \begin{bmatrix} \delta_{ij} & -v^i \\ -v^j & -1 + v^k v^k \end{bmatrix}, \quad \det p_{\mu\nu} = -1. \tag{4.31}$$

Let $\left\{{\varrho \atop \mu\nu}\right\}_p$ denote the Christoffel symbols based on the tensor $p_{\mu\nu}$. Except for special motions, these Christoffel symbols define a world connection in Galilean space-time which is distinct from the Galilean connection. The Riemannian curvature tensor based on the $\left\{{\varrho \atop \mu\nu}\right\}_p$ does not vanish except for special motions. The Galilean connection is independent of a motion; the Christoffel symbols based on $p_{\mu\nu}$ are a type of differential invariant of a motion in $\mathscr{S}_{\mathrm{E}}$ or $\mathscr{S}_{\mathrm{G}}$. Before proceeding, we list some invariant algebraic relations satisfied by $p_{\mu\nu}$:

$$p^{\mu\nu} t_\nu = -v^\mu, \qquad p_{\mu\nu} v^\nu = -t_\mu, \qquad p_{\mu\varrho} p_{\nu\lambda} g^{\varrho\lambda} = p_{\mu\nu} + t_\mu t_\nu. \tag{4.32}$$

The canonical form of the Christoffel symbols is

$$\begin{gathered}\left\{{i \atop jk}\right\}_p = v^i d^{jk}, \quad \left\{{i \atop j4}\right\}_p = -\omega^{ij} - v^i v^s d^{sj}, \quad \left\{{i \atop 44}\right\}_p = -\frac{\partial v^i}{\partial z^4} - v^s d^{si} - v^s \omega^{si} + v^i v^s v^r d^{sr}, \\ \left\{{4 \atop 4i}\right\}_p = -v^s d^{is}, \quad \left\{{4 \atop 44}\right\}_p = v^r v^s d^{rs}, \quad \left\{{4 \atop ij}\right\}_p = d^{ij},\end{gathered} \tag{4.33}$$

where $\omega^{ij} \equiv \frac{1}{2}(\partial_i v^j - \partial_j v^i)$.

Since adding any world tensor $S^\varrho_{\mu\nu} = S^\varrho_{\nu\mu}$ to the Christoffel symbols yields another set of quantities transforming as the components of a symmetric world connection [*1*, p. 6], we see that one can construct an infinity of world connections all of which are part of the Euclidean and Galilean geometry of a motion. All that is needed is to be able to define a tensor of rank 3, $S^\varrho_{\mu\nu}$, in terms of the motion, $g^{\mu\nu}$ and t_μ. The tensor $p_{\mu\nu} v^\varrho$ is one such admissible tensor, there being an infinite number of others. Without further motivation derived from physical applications or intuition, there seems little to recommend an intensive study of this variety of differential invariants of a motion. Defrise [*20*] has based the determination of a world connection in Euclidean space-time upon the intuitive notion that the world lines of the particles of the medium shall be a system of "parallel straight lines" in the 4-dimensional sense, plus other ideas based on the parallel transport of tensor fields. Having determined such a connection, Defrise proceeds to study in detail various other invariants of a motion that can be defined with his connection, such as its Riemann tensor. The equations used by Defrise to determine a world connection $\Omega^\varrho_{\mu\nu}$ (hereafter referred to as the Defrise connection) are equivalent to the following:

$$\underset{\Omega}{\nabla_\nu} v^\mu = 0, \qquad \underset{\Omega}{\nabla_\lambda} g^{\mu\nu} = 2\Delta^{\mu\nu} t_\lambda, \qquad \underset{\Omega}{\nabla_\nu} t_\mu = 0, \tag{4.34}$$

where the operator $\underset{\Omega}{\nabla_\mu}$ denotes covariant differentiation based on the Defrise connection $\Omega^\varrho_{\mu\nu}$. These equations have a unique solution for all 64 components of $\Omega^\varrho_{\mu\nu}$ in terms of the fields $g^{\mu\nu}$, t_λ, v^μ and their derivatives. The canonical form of the Defrise connection is

$$\Omega^i_{jk} = 0, \qquad \Omega^4_{\mu\nu} = 0, \qquad \Omega^i_{4j} = -\partial_j v^i, \qquad \Omega^i_{44} = -\frac{\partial v^i}{\partial z^4} + v^j \partial_j v^i. \tag{4.35}$$

The difference between the Defrise connection and the Christoffel symbols based on $p_{\mu\nu}$ is a world tensor given by

$$S^\varrho_{\mu\nu} \equiv \left\{{\varrho \atop \mu\nu}\right\}_p - \Omega^\varrho_{\mu\nu} = p_{\mu\lambda} p_{\nu\zeta} \Delta^{\lambda\zeta} v^\varrho + p_{\mu\lambda} \Delta^{\lambda\varrho} t_\nu + p_{\nu\lambda} \Delta^{\lambda\varrho} t_\mu. \tag{4.36}$$

The canonical form of $S^{\varrho}_{\mu\nu}$ is

$$S^i_{jk} = v^i d^{jk}, \quad S^i_{j4} = -v^i v^s d^{js}, \quad S^4_{4i} = -v^s s^{is}, \quad S^4_{ij} = d^{ij}, \\ S^i_{44} = v^i v^s v^r d^{sr} - 2v^s d^{is}, \qquad S^4_{44} = v^r v^s d^{rs}. \tag{4.37}$$

As a variant of DEFRISE's procedure, we can solve the equations

$$\underset{\Psi}{\nabla}_\lambda g^{\mu\nu} = 0, \quad v^\nu \underset{\Psi}{\nabla}_\nu v^\mu = 0, \quad \underset{\Psi}{\nabla}_\nu t_\mu = 0, \quad \underset{\Psi}{\nabla}_\lambda v^{[\mu} g^{\nu]\lambda} = 0 \tag{4.38}$$

for the components of yet another symmetric world connection $\Psi^{\varrho}_{\mu\nu}$. According to $(4.38)_2$, the world lines of the material points undergoing the motion will be paths of the connection $\Psi^{\varrho}_{\mu\nu}$ [*1*, § 22]. The canonical form of this connection is

$$\Psi^i_{jk} = 0, \quad \Psi^4_{\mu\nu} = 0, \quad \Psi^i_{4j} = \omega^{ij}, \quad \Psi^i_{44} = (\omega^{si} - d^{si})\, v^s - \frac{\partial v^i}{\partial z^4}, \tag{4.39}$$

and the difference between this connection and the Defrise connection is the world tensor $U^{\varrho}_{\mu\nu}$ given by

$$U^{\varrho}_{\mu\nu} \equiv \Psi^{\varrho}_{\mu\nu} - \Omega^{\varrho}_{\mu\nu} = p_{\mu\lambda} \Delta^{\lambda\varrho} t_\nu + p_{\nu\lambda} \Delta^{\lambda\varrho} t_\mu. \tag{4.40}$$

The formulae (4.36) and (4.40) are convenient to have when one considers a question of the type: What is the invariant significance of the absolute derivative of the velocity field with respect to the Christoffel symbols $\left\{{\varrho \atop \mu\nu}\right\}_p$? The answer follows simply from the result (4.36). Let $\underset{p}{\nabla}_\mu$ denote covariant differentiation based on the $\left\{{\varrho \atop \lambda\mu}\right\}_p$. We then have

$$\underset{p}{\nabla}_\nu v^\mu = \partial_\nu v^\mu + \left\{{\mu \atop \nu\varrho}\right\}_p v^\varrho = \partial_\nu v^\mu + \Omega^\mu_{\nu\varrho} v^\varrho + S^\mu_{\nu\varrho}. \tag{4.41}$$

The first two terms on the right hand side of (4.41) cancel each other by $(4.34)_1$. Substituting from (4.36) for the $S^\mu_{\nu\varrho}$ yields

$$\underset{p}{\nabla}_\nu v^\mu = p_{\nu\lambda} \Delta^{\mu\lambda}. \tag{4.42}$$

If we multiply this equation by $p^{\nu\varrho}$ and sum, we get

$$v^{\varrho\mu} = p^{\nu\varrho} \underset{p}{\nabla}_\nu v^\mu = \Delta^{\varrho\mu}. \tag{4.43}$$

Since $\Delta^{\mu\nu}$ satisfies the invariant equation $\Delta^{\mu\nu} t_\nu = 0$, we have from (4.42)

$$v^\nu \underset{p}{\nabla}_\nu v^\mu = v^\nu p_{\nu\lambda} \Delta^{\mu\lambda} = -t_\lambda \Delta^{\mu\lambda} = 0. \tag{4.44}$$

Thus *the world lines of the motion are paths of the connection* $\left\{{\varrho \atop \mu\nu}\right\}_p$.

Let $\underset{\Gamma}{\nabla}_\mu$ denote covariant differentiation based on the Galilean connection $\Gamma^{\varrho}_{\mu\nu}$. The *world acceleration vector* of a motion in Galilean space-time is defined by

$$\alpha^\mu \equiv v^\nu \underset{\Gamma}{\nabla}_\nu v^\mu. \tag{4.45}$$

The canonical form of the world acceleration vector is

$$\alpha^\mu = (a^i, 0), \qquad a^i = \frac{\partial v^i}{\partial z^4} + v^j \partial_j v^i. \tag{4.46}$$

The transformation law relating the components a^i and $a^{i'}$ in two Galilean frames is

$$a^{i'} = A^{i'}_j a^j, \tag{4.47}$$

where $A^{i'}_j$ is a constant orthogonal matrix. Note that the world acceleration vector of a motion in Euclidean space-time is undefined.

Consider the world tensor $W^{\mu\nu}$ defined by

$$W^{\mu\nu} \equiv g^{\mu\lambda} \underset{\Gamma}{\nabla_\lambda} v^\nu. \tag{4.48}$$

The canonical form of this tensor is

$$W^{\mu\nu} = \begin{bmatrix} w^{ij} & 0 \\ 0 & 0 \end{bmatrix}, \qquad w^{ij} = \partial_i v^j. \tag{4.49}$$

The symmetric and antisymmetric parts of $W^{\mu\nu}$ are world tensors

$$\Delta^{\mu\nu} = W^{(\mu\nu)}, \qquad \Omega^{\mu\nu} = W^{[\mu\nu]}, \qquad W^{\mu\nu} = \Delta^{\mu\nu} + \Omega^{\mu\nu}, \tag{4.50}$$

where the canonical form of $\Omega^{\mu\nu}$ is

$$\Omega^{\mu\nu} = \begin{bmatrix} \omega^{ij} & 0 \\ 0 & 0 \end{bmatrix}, \qquad \omega^{ij} = \tfrac{1}{2}\left(\partial_i v^j - \partial_j v^i\right). \tag{4.51}$$

Since ω^{ij} is the classical measure of vorticity, we call $\Omega^{\mu\nu}$ the *world vorticity tensor*. Now the world rate of deformation tensor $\Delta^{\mu\nu}$ was defined in (4.16) independently of the Galilean connection; whereas, from (4.50) it is not immediately apparent that the symmetric part of $W^{\mu\nu}$ can be expressed solely in terms of the fields v^μ, $g^{\mu\nu}$ and the derivatives of these fields. That this is the case follows from $(3.16)_1$.

$$W^{(\mu\nu)} = g^{\lambda(\mu} \partial_\lambda v^{\nu)} + g^{\lambda(\mu} \Gamma^{\nu)}_{\lambda\varrho} v^\varrho, \tag{4.52}$$

and from $(3.16)_1$ we get

$$2 g^{\lambda(\mu} \Gamma^{\nu)}_{\lambda\varrho} = - \partial_\varrho g^{\mu\nu}. \tag{4.53}$$

Substituting this last result into (4.52) we verify the identity $W^{(\mu\nu)} \equiv -\frac{1}{2} \underset{v}{\pounds} g^{\mu\nu}$. The components of the world vorticity tensor $\Omega^{\mu\nu}$ cannot in this same way be expressed solely in terms of the fields v^μ, $g^{\nu\pi}$, t_σ and their derivatives. That is, vorticity is not a Euclidean invariant of a motion. But this is clear from an intuitive point of view since vorticity measures a rate of rotation and rotation does not have an invariant significance under the Euclidean group, but does have an invariant significance under the smaller Galilean group.

5. Lorentzian kinematics

The study of invariants of a motion in Lorentzian space-time has important applications in relativistic mechanics and electromagnetic theory. The symmetrical treatment of all four coordinates for events has found greater usage and favor in relativity theory than in classical mechanics. However, as we have seen, the formulation of classical invariant-theoretic problems in terms of a 4-dimensional geometry is easily accomplished and has some formal manipulative advantages. There appears still to exist some misunderstanding in applied work concerning the introduction of general coordinates in space-time and the physical implications of such a process. The equations of classical mechanics, as well as all of the purely kinematical considerations we have given here, can be phrased and presented in terms of world invariants of a suitable set of fields in space-time.

This does not alter the physical hypotheses or interpretations constituting the theory any more or less than the familiar use of curvilinear coordinates for positions in space. McVittie [5] has considered certain of the Galilean invariants of a motion as limiting cases of Lorentz invariants with infinitesimal velocities. Here we have preferred an independent development of each of these theories of motion treating each as an exact science. One of the best sources for ideas and results in Lorentzian kinematics is Møller's book on relativity theory [9]. Following is a brief discussion of a few Lorentz invariants of a motion whose importance, relative to others we might consider, is suggested by applications in relativistic theories of elastic bodies and fluids.

Consider the world axial vector field of weight 1 defined by

$$\mathfrak{v}^\mu \equiv \frac{1}{3!}\,\varepsilon^{\varrho\nu\lambda\mu}\,\partial_\varrho \overset{K}{Z}\,\partial_\nu \overset{L}{Z}\,\partial_\lambda \overset{M}{Z}\,\varepsilon_{KLM}. \tag{5.1}$$

The world scalar of weight 2 given by

$$\boldsymbol{\theta} \equiv \gamma_{\mu\nu}\,\mathfrak{v}^\mu\,\mathfrak{v}^\nu \tag{5.2}$$

is always *negative.* This follows on substituting the definition of $\mathfrak{v}^\mu$ into (5.2) and deriving the identity

$$\boldsymbol{\theta} = \det \gamma_{\mu\nu} \det \overset{KL}{X^{-1}} < 0. \tag{5.3}$$

The inequality holds since the determinant of the Lorentz metric is always negative and the determinant of $\overset{KL}{X^{-1}}$ is by assumption always positive. The absolute world vector defined by

$$w^\mu \equiv \frac{\mathfrak{v}^\mu}{\sqrt{-\boldsymbol{\theta}}}, \qquad \gamma_{\mu\nu}\,w^\mu\,w^\nu = -1, \tag{5.4}$$

is called the *relativistic world velocity vector of the motion.*

Let the motion be referred to an arbitrary Lorentz frame. We then have

$$\begin{aligned} \overset{KL}{X^{-1}} &= \delta^{ij}\,\partial_i \overset{K}{Z}\,\partial_j \overset{L}{Z} - \partial_4 \overset{K}{Z}\,\partial_4 \overset{L}{Z}, \\ \delta^{ij}\,\partial_i \overset{K}{Z}\,\partial_j \overset{L}{Z} &= \overset{KL}{X^{-1}} + \partial_4 \overset{K}{Z}\,\partial_4 \overset{L}{Z}. \end{aligned} \tag{5.5}$$

Taking the determinant of both sides of this last equation yields

$$\mathfrak{D}^2 = \left(\det \partial_i \overset{K}{Z}\right)^2 = \det \overset{KL}{X^{-1}}\left(1 + \underset{KL}{X}\,\partial_4 \overset{K}{Z}\,\partial_4 \overset{L}{Z}\right) > 0, \tag{5.6}$$

where $\underset{KL}{X}$ is the inverse of the matrix $\overset{KL}{X^{-1}}$. Since $\overset{KL}{X^{-1}}$ is positive definite, so is its inverse, and from (5.6) we see that $\mathfrak{D}^2$ is never zero for any motion of a material medium in Lorentzian space-time. This means that we can always solve for the first three Lorentz coordinates z^i of a Lorentz frame in terms of the Z^K and z^4

$$z^i = \overset{i}{z}(Z^K, z^4). \tag{5.7}$$

The canonical form of the axial vector $\mathfrak{v}^\mu$ is

$$\mathfrak{v}^\mu = \mathfrak{D}\left(\frac{\partial \overset{i}{z}}{\partial z^4}, 1\right) = \mathfrak{D}(v^i, 1), \tag{5.8}$$

where v^i is the "classical velocity" of a material point relative to the Lorentz frame z^μ, where we think of z^4 as classical time. This is not a Lorentz invariant notion since z^4 and T have different transformation laws. The canonical form of (5.2) is

$$\theta = \mathfrak{D}^2 (v^i v^i - 1) < 0, \tag{5.9}$$

where the inequality follows from (5.3). Thus we see that the hypothesis that $\overset{KL}{X^{-1}}$ is positive definite for the motion of a material medium in Lorentzian space-time leads to the familiar result that the "classical velocity" of the motion relative to *any* Lorentz frame is always less than 1, where this upper limit for the speed of any motion is identified with the speed of light in the chosen system of units.

The components of the relativistic world velocity vector w^μ in a Lorentz frame have the values

$$w^\mu = \left(\frac{v^i}{\sqrt{1-v^2}}, \frac{1}{\sqrt{1-v^2}}\right), \qquad v^2 \equiv v^i v^i. \tag{5.10}$$

In relativistic mechanics, the fourth component of w^μ in a Lorentz frame less 1 is called the *kinetic energy per unit of mass* [*4*].

An affine connection in $\mathscr{S}_{\mathrm{L}}$ is determined by the Christoffel symbols based on the Lorentz metric. All of these symbols vanish in a Lorentz frame. Let $\underset{\gamma}{\nabla}_\mu$ denote covariant differentiation based on the Christoffel symbols of $\gamma_{\mu\nu}$. The *world velocity gradients* are defined by $\underset{\gamma}{\nabla}_\mu w^\nu$. The relativistic counterparts of the world rate of deformation tensor and the world vorticity tensor of Euclidean and Galilean kinematics can be defined as follows:

$$\underset{\mathrm{L}}{\varDelta}^{\mu\nu} \equiv \gamma^{\lambda(\mu} \underset{\gamma}{\nabla}_\lambda w^{\nu)}, \qquad \underset{\mathrm{L}}{\varOmega}^{\mu\nu} \equiv \gamma^{\lambda[\mu} \underset{\gamma}{\nabla}_\lambda w^{\nu]}, \tag{5.11}$$

where we shall have the identity

$$\underset{\mathrm{L}}{\varDelta}^{\mu\nu} \equiv -\tfrac{1}{2} \underset{w}{\pounds}\, \gamma^{\mu\nu}, \tag{5.12}$$

analogous to the classical case (4.16), (4.50).

In the case of Euclidean or Galilean kinematics, a rigid motion can be defined by either of the conditions $v^\mu \partial_\mu \overset{KL}{C^{-1}} = 0$, or $\varDelta^{\mu\nu} = 0$. In the case of a motion in Lorentzian space-time, however, the two analogous conditions

$$w^\mu \partial_\mu \overset{KL}{X^{-1}} = 0, \qquad \underset{\mathrm{L}}{\varDelta}^{\mu\nu} = 0 \tag{5.13}$$

are *not* equivalent. There exist motions for which we have $(5.13)_1$ and do not have $(5.13)_2$. In fact, if w^μ satisfies $(5.13)_2$, then it is a *translation* [*10*, p. 349]. Its components in a Lorentz frame will have the form (5.10), where the v^i will be constants. The motion

$$\overset{K}{Z}(z^i, z^4) = A_i^K(z^4)\, z^i, \qquad A_i^K A_i^L = \delta^{KL}, \tag{5.14}$$

representing a rotation about the origin of the Lorentz frame z^i is an admissible motion in $\mathscr{S}_{\mathrm{L}}$ for all values of z^i and z^4 such that

$$\overset{KL}{X^{-1}} = \delta^{KL} - \dot{A}_i^K \dot{A}_j^L z^i z^j \tag{5.15}$$

is positive definite. This will be true for all sufficiently small values of z^i. If the time derivatives of the $\overset{K}{\underset{i}{A}}$ are constants, then the first of the conditions (5.13) is satisfied. The motion (5.14) is a rigid motion in Euclidean or Galilean space-time. Thus it is reasonable to define a rigid motion in Lorentzian space-time by the first of the conditions (5.13). We are aware that there has been considerable debate as to what a useful and appropriate definition of a rigid motion in relativistic kinematics should be. We see that the conditions $(5.13)_2$ would be too restrictive since they rule out all but the uniform translations. Pursuing the analogy with Euclidean kinematics, consider the functions $(X^{-1})^{KL}(Z^K, z^4)$ defined by

$$(X^{-1})^{KL}(Z^K, z^4) \equiv \overset{KL}{X^{-1}}\left(z^i(Z^K, z^4), z^4\right), \tag{5.16}$$

where the functions $\overset{i}{z}$ are those occurring in (5.7). Thus, treating z^4 as a parameter, we can construct the Riemann curvature tensor $R_{\dot{L}\dot{M}\dot{N}}{}^K(X_{KL})$, $X_{KL}(X^{-1})^{LM} = \delta^M_K$. MØLLER [9] calls a motion in $\mathscr{S}_{\mathrm{L}}$ satisfying the conditions

$$R_{\dot{L}\dot{M}\dot{N}}{}^K(X_{PQ}) = 0 \tag{5.17}$$

a *Euclidean motion.* The rigid motion (5.14) is *not* a Euclidean motion. Equation (5.17) is to be compared with its classical counterpart (4.12), which holds for *any* motion in $\mathscr{S}_{\mathrm{E}}$ or $\mathscr{S}_{\mathrm{G}}$. Moreover, (4.12) is a Euclidean and Galilean invariant property of any motion in $\mathscr{S}_{\mathrm{E}}$ or $\mathscr{S}_{\mathrm{G}}$; whereas, if (5.17) is satisfied by a motion relative to one Lorentz frame, it need not be satisfied by the same motion referred to another Lorentz frame. That is, since z^4 is not an invariant parameter in (5.17) under Lorentz transformations, this condition is not Lorentz invariant.

In special relativity theory, the equations representing conservation of energy, conservation of momentum, conservation of angular momentum, and the equivalence of momentum and energy flux take the form [4]

$$\underset{\gamma}{\nabla_\nu} P^{\mu\nu} = 0, \qquad P^{\mu\nu} = P^{\nu\mu}, \tag{5.18}$$

where $P^{\mu\nu}$ is the stress-energy-momentum tensor. It is customary to write $P^{\mu\nu}$ in the form

$$P^{\mu\nu} = \tau^{\mu\nu} - \varrho\, w^\mu w^\nu, \tag{5.19}$$

where $\tau^{\mu\nu}$ embodies the relativistic counterparts of the classical stress, internal energy, and heat flux. In classical mechanics, it is customary to provide constitutive equations for all of these quantities in terms of the motion, the temperature, the electromagnetic field, *etc.*, and to require their invariance under various groups of transformations such as the rigid motions. But in relativity theory, owing to the open question of a proper definition of a rigid motion which is not a uniform translation, a clear and concise statement of the class of admissible constitutive equations for $\tau^{\mu\nu}$ depending on a motion cannot, to our knowledge, be found in the literature. *The relativistic counterpart of the fundamental assumption* (1.11) *of classical continuum mechanics has not been enunciated.* It would seem that a sound relativistic generalization of, say, classical elasticity theory rests on questions of this nature.

6. Convected coordinates in Euclidean and Galilean space-time

Given a motion in Galilean or Euclidean space-time, there are, in addition to the Galilean and Euclidean frames, other classes of preferred coordinates defined in a natural way in terms of the motion [*12, 14*]. One such class of coordinate systems are the convected coordinates belonging to a motion [*11, 7*]. A convected coordinate system of a motion in $\mathcal{S}_E$ or $\mathcal{S}_G$ can be defined as any coordinate system x^Ω (we use upper case Greek indices to indicate convected coordinates) in which the world velocity vector v^μ and the covariant space normal t_μ have the *convected form*

$$v^\Omega = (0, 0, 0, 1), \qquad t_\Omega = (0, 0, 0, 1). \tag{6.1}$$

The existence of such coordinate systems is easy to perceive. Let $\overset{K}{X}(Z^L)$ be any three functionally independent functions of the Z^L. If we transform the components v^μ and t_μ in an arbitrary system of coordinates x^μ to the coordinate system x^Ω determined by

$$\begin{aligned} x^K &= \overset{K}{X}\big(\overset{L}{Z}(x^\mu)\big), \\ x^4 &= t(x^\mu) = T, \end{aligned} \tag{6.2}$$

we conclude immediately that the coordinate system (x^K, T) is a convected coordinate system. Any two convected coordinate systems x^Ω and $x^{\Omega'}$ are related by a transformation having the general form

$$\begin{aligned} x^{K'} &= \overset{K'}{x}(x^K), \\ x^{4'} &= x^4 + \text{constant}. \end{aligned} \tag{6.3}$$

From $(3.5)_1$ it follows immediately that in a convected frame

$$g^{\Omega 4} = g^{4\Omega} = 0, \tag{6.4}$$

and from the definition of the scalars $\overset{KL}{C^{-1}}$ we have

$$\overset{KL}{C^{-1}} = g^{MN} \frac{\partial \overset{K}{Z}}{\partial x^M} \frac{\partial \overset{L}{Z}}{\partial x^N}. \tag{6.5}$$

If we consider the special case in which $\overset{K}{X} = \overset{K}{Z}$, we have the simpler relation

$$\overset{KL}{C^{-1}} = g^{KL}. \tag{6.6}$$

Thus *the non-vanishing components of $g^{\mu\nu}$ in a convected frame $x^K = Z^K$ of a motion are equal respectively to the six material measures of deformation* $\overset{KL}{C^{-1}}$. Some workers in elasticity theory [*11*] who employ convected coordinates almost exclusively refer to the quantities C_{KL} as the metric tensor. The result (6.6) provides the principal motivation for the use of such terminology. However, it must be realized that the equality (6.6) holds only in a restricted class of coordinate systems.

Consider next the rate of deformation tensor $\Delta^{\mu\nu}$, whose components in a general frame are given by (4.16). Substituting the convected form of the velocity vector into this formula, we conclude that

$$\Delta^{\Omega 4} = \Delta^{4\Omega} = 0, \qquad \Delta^{KL} = -\frac{1}{2}\frac{\partial g^{KL}}{\partial T}. \tag{6.7}$$

More generally, we have for the series of world tensors $\overset{*}{\Delta}{}^{\mu\nu}, \ldots, \overset{(n)}{\Delta}{}^{\mu\nu}, \ldots$ [*cf.* (4.27)]

$$\overset{(n)}{\Delta}{}^{\Omega 4} = \overset{(n)}{\Delta}{}^{4\Omega} = 0, \quad \overset{(n)}{\Delta}{}^{KL} = -\frac{1}{2}\frac{\partial^n g^{KL}}{\partial T^n}. \tag{6.8}$$

Since the covariant derivatives of $g^{\mu\nu}$ and t_μ with respect to the Galilean connection vanish in an arbitrary coordinate system, we have

$$\partial_\Theta t_\Omega - \Gamma^{\Psi}_{\Theta\Omega} t_\Psi = 0, \tag{6.9}$$

$$\partial_\Psi g^{\Theta\Omega} + \Gamma^{\Omega}_{\Psi\Delta} g^{\Delta\Theta} + \Gamma^{\Theta}_{\Psi\Delta} g^{\Omega\Delta} = 0. \tag{6.10}$$

From (6.9) and $(6.1)_2$ it follows that, in a convected frame,

$$\Gamma^4_{\Omega\Theta} = 0. \tag{6.11}$$

From (6.10) we obtain the equations

$$\Gamma^K_{LM} = \left\{ {K \atop LM} \right\}_g, \quad \Gamma^{(K}_{4M} g^{L)M} = -\frac{1}{2}\frac{\partial g^{KL}}{\partial T} = \Delta^{KL}, \tag{6.12}$$

where the Γ^K_{LM} are the Christoffel symbols based on g_{KL}, $g_{KL} g^{LM} = \delta^M_K$.

The components of the world vorticity tensor $\Omega^{\Phi\Psi}$ in a convected frame are given by

$$\Omega^{\Theta 4} = -\Omega^{4\Theta} = 0, \quad \Omega^{KL} = g^{M[K}\Gamma^{L]}_{M4}. \tag{6.13}$$

The world acceleration vector has components given by

$$a^4 = 0, \quad a^K = \Gamma^K_{44}. \tag{6.14}$$

The formulae of this section are useful for the interpretation of any system of deforming and accelerated coordinates in Euclidean or Galilean space-time not necessarily associated with the motion of a material medium. That is, consider the class of all coordinate systems in $\mathscr{S}_E$ or $\mathscr{S}_G$ in which we have $t_\mu = (0, 0, 0, 1)$. Equation (6.7) then gives an interpretation of the time dependence of the non-vanishing components of the space metric $g^{\mu\nu}$ in one of these general types of coordinate systems. $-\frac{1}{2}\frac{\partial g^{KL}}{\partial T}$ is a measure of the rate of deformation of the coordinate system. Similarly, the formulae (6.12), (6.13), and (6.14) provide an interpretation of the non-vanishing components of the Galilean connection in such a system of coordinates.

7. The geometry of the space of material points and material symmetry

In continuum mechanics we assign a geometry to the 3-dimensional space $\mathscr{S}_M$ with coordinates Z^K by introducing a group $\mathfrak{G}_M$ of *allowable material coordinate transformations*. A typical element of this group has the form

$$Z^{K'} = A^{K'}_L Z^L + D^{K'}, \tag{7.1}$$

where, in all of the applications with which we are familiar, it is sufficiently general to assume that $\mathfrak{G}_M$ is some subgroup of the 3-dimensional orthogonal group. Thus the matrix of coefficients in (7.1) is an *orthogonal matrix*. By demanding invariance of constitutive equations under $\mathfrak{G}_M$ we obtain further restrictions on the form of these equations. The group $\mathfrak{G}_M$ determines the *material*

symmetry of the medium. Let us see how this relation between the group of material coordinate transformations $\mathfrak{G}_M$ and the idea of material symmetry arises. It is customary to require that, for some instant T_0, the functions $\overset{K}{Z}(z^i, T)$ representing a motion relative to a Euclidean or Galilean frame (z^i, T) reduce to the form

$$z^i = \delta^i_K \overset{K}{Z}(z^i, T_0) = \delta^i_K Z^K. \tag{7.2}$$

That is, the material coordinates Z^K at the instant T_0 coincide with the Cartesian coordinates z^i. Now if the material points are all identical and arranged in space in a uniform homogeneous array, the intuitive notion is that we cannot in this way ascribe unique names to each material point, but that the names (material coordinates Z^K) are determined only to within an arbitrary orthogonal transformation. This is the intuitive picture of an *isotropic homogeneous material medium*. Thus the appropriate group $\mathfrak{G}_M$ for a material with this symmetry is the *complete orthogonal group*. If some of the material points have different properties than others, such as in a crystal, and if they are arranged in space in some non-uniform or inhomogeneous array at the instant T_0, the class of equivalent (allowable) material coordinate systems Z^K will be smaller than the corresponding class for isotropic homogeneous materials. Thus the group $\mathfrak{G}_M$ in the general case will be some proper subgroup of the orthogonal group. The continuum theory of elastic homogeneous crystals is based on constitutive equations for the stress which are invariant under a group of material coordinate transformations $\mathfrak{G}_M$, where the set of matrices $A^{K'}_L$ in (7.1) constitute the elements of one of the 32 crystallographic subgroups of the orthogonal group characteristic of the *point* symmetry of the crystal. In the theory of finite elastic deformations of crystals and in the classical linear theory of elastic crystals the constitutive equations are assumed invariant to arbitrary translations D^K in (7.1). This last assumption represents an approximation to a more detailed description of the symmetry of a crystal in which one would require invariance only to a group of discrete translations D^K. However, the present formalism does not rule out the possibility of an accurate and detailed continuum description of such "microscopic" structure or symmetry. Other choices for the group $\mathfrak{G}_M$ describe materials having transverse isotropic symmetry, orthotropic symmetry, *etc.*, [*19*].

Let us illustrate the way in which the invariance of constitutive equations under the group $\mathfrak{G}_M$ restricts their general form by considering the case of the stress tensor in finite elasticity theory. In § 1 we remarked that if the stress tensor t^{ij} in an elastic body of any symmetry whatever were invariant under the group of rigid motions, then it must reduce to the form

$$t^{ij} = \frac{\partial \overset{i}{x}}{\partial Z^K} \frac{\partial \overset{j}{x}}{\partial Z^L} \overset{KL}{P}(\overset{M}{Z}, \overset{PQ}{C^{-1}}, \mathfrak{D}). \tag{7.2 a}$$

If we now demand that t^{ij} transform as an absolute scalar under the group $\mathfrak{G}_M$ of all orthogonal transformations, whereby we assume that the material is isotropic and homogeneous, then it is known that the functions $\overset{KL}{P}$ are expressible in the special form [*12*]

$$\overset{KL}{P} = \mathscr{F}_0 \delta^{KL} + \mathscr{F}_1 \overset{KL}{C^{-1}} + \mathscr{F}_2 \overset{KL}{C^{-2}}, \tag{7.3}$$

where the $\mathscr{F}$'s are functions of the scalar invariants of the matrix $\overset{KL}{C^{-1}}$.

$$\mathscr{F}_\Omega = \mathscr{F}_\Omega(\mathrm{I}, \mathrm{II}, \mathrm{III}), \qquad \Omega = 0, 1, 2,$$
$$\mathrm{I} \equiv \operatorname{trace} \overset{KL}{C^{-1}}, \ \mathrm{II} \equiv \text{sum of the principal minors of } \overset{KL}{C^{-1}}, \ \mathrm{III} \equiv \det \overset{KL}{C^{-1}} \equiv \mathfrak{D}^2. \tag{7.4}$$

The constitutive equations for the stress in a material with less symmetry than an isotropic homogeneous medium will involve functions $\overset{KL}{P}$ of $\overset{K}{Z}$, $\overset{KL}{C^{-1}}$, and $\mathfrak{D}$ more complicated than (7.3).

In presenting these few remarks and examples concerning the relation between material symmetry and the invariant-theoretic problems encountered in the formulation of constitutive relations in continuum mechanics, we have attempted to make clear that the problem is one of invariance under at least two distinct groups of transformations: 1) invariance to rigid motions, 2) invariance under a group of material coordinate transformations. A third group not considered here is the group of *unit* transformations. It is important that these three demands for invariance not be confused and that each of them be satisfied. A confusion of this sort is the apparent source of difficulty in some recent attacks on the foundations of classical elasticity theory [*18*, *et al.*]. As we have seen, these invariances are not equivalent.

In the 3-dimensional space $\mathscr{S}_\mathrm{M}$ with group $\mathfrak{G}_\mathrm{M}$ we can introduce general coordinates X^K and appeal once again to KLEIN's principle so as to obtain an equivalent statement of the invariant theoretic problem. Let $\mathfrak{G}_{\mathrm{MA}}$ denote the group of unrestricted analytic transformations on the material coordinates X^K with typical element*

$$X^{K'} = \overset{K'}{Y}(X^K), \qquad X^K = \overset{K}{Y}(X^{K'}). \tag{7.5}$$

Let $\boldsymbol{H}_\Omega(X^K)$, $\Omega = 1, 2, \ldots, N$ be a set of fields having an assigned law of transformation under $\mathfrak{G}_{\mathrm{MA}}$ such that the fields $\boldsymbol{H}_\Omega$ possess certain canonical forms $\boldsymbol{H}_\Omega(Z^K)$ in a subclass Z^K of the general coordinate systems X^K and such that the group $\mathfrak{G}_\mathrm{M}$ consists in all the transformations of $\mathfrak{G}_{\mathrm{MA}}$ which leave these canonical forms invariant. By KLEIN's principle, the geometry of a field or set of fields such as $C_{KL}(Z^M, T)$ with respect to the group $\mathfrak{G}_\mathrm{M}$ is equivalent to the geometry of the fields $C_{KL}(X^M, T)$ together with the fields $\boldsymbol{H}_\Omega(X^K)$ under the group $\mathfrak{G}_{\mathrm{MA}}$.

Since we assume that $\mathfrak{G}_\mathrm{M}$ is some subgroup of the orthogonal group, we can always choose for one of the fields $\boldsymbol{H}_\Omega$ a symmetric positive definite Euclidean metric tensor $G_{KL}(X^M)$ and identify the frames Z^K with some subclass of the coordinate systems in which we have $G_{KL}(Z^M) = \delta_{KL}$. If $\mathfrak{G}_\mathrm{M}$ is the complete orthogonal group, G_{KL} is the only field in the set $\boldsymbol{H}_\Omega$ since the orthogonal group consists in all the transformations which leave its canonical form δ_{KL} invariant. The procedure of introducing general coordinates in the space $\mathscr{S}_\mathrm{M}$ of an isotropic homogeneous material can be illustrated by writing down the form of (7.3) which is invariant under $\mathfrak{G}_{\mathrm{MA}}$. It is as follows [*8*]:

$$P^{KL}(X^M, T) = \mathscr{F}_0 G^{KL}(X^M) + \mathscr{F}_1 (C^{-1})^{KL}(X^M, T) + \mathscr{F}_2 (C^{-2})^{KL}, \tag{7.6}$$

---

* Here we must use a different symbol for the functions $\overset{K}{Y}$ representing a material coordinate transformation so as not to confuse these functions of 3 variables with the motion $\overset{K}{X}(x^\mu)$ represented by functions of 4 variables x^μ.

where we put $(C^{-1})^K_L = G_{LM}(C^{-1})^{KM}$, $(C^{-2})^{KL} = G^{KM}(C^{-1})^L_N(C^{-1})^N_M$. The $\mathcal{F}$'s are functions of I, II, and III, now given by

$$\mathrm{I} = (C^{-1})^K_K, \quad \mathrm{II} = \frac{1}{2!}\,\delta^{KL}_{PR}(C^{-1})^P_K(C^{-1})^R_L, \quad \mathrm{III} = \frac{1}{3!}\,\delta^{KLM}_{PRS}(C^{-1})^P_K(C^{-1})^R_L(C^{-1})^S_M. \tag{7.7}$$

8. Two-point tensor fields and world invariant kinematics

We have seen that in continuum mechanics we are interested in functions of a motion such as the stress tensor in elasticity theory which are invariant under two groups of transformations. The first of these groups involves transformations of the coordinates of points (events) in a 4-dimensional space and the second of these groups involves transformations of the coordinates of points (material points) in a 3-dimensional space. *Multiple point fields* are familiar objects in pure geometry and have been used to advantage in continuum mechanics [*2, 19, 17*]. The concept of a 2-point field is an easy generalization of the concept of a 1-point field set forth in § 2. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two spaces, not necessarily of the same dimension, with groups $\mathfrak{G}_1$ and $\mathfrak{G}_2$, respectively. Let X^Ω, $\Omega = 1, 2, \ldots, N$ denote coordinates for the points in $\mathcal{S}_1$ and let x^μ, $\mu = 1, 2, \ldots, n$ denote coordinates for points in $\mathcal{S}_2$. A *2-point field* is a set of functions $F_\Delta(X^\Omega, x^\mu)$ of the coordinates of a point in $\mathcal{S}_1$ and of a point in $\mathcal{S}_2$. Thus the components F_Δ of a 2-point field will depend in general on $n+N$ independent variables. A 2-point field has a law of transformation for its components under independent transformations of the coordinates of either point. The law of transformation may be any law consistent with the property that a representation $F_\Delta(X^\Omega, x^\mu)$ in any system of reference (X^Ω, x^μ) determines a unique representation $F_\Delta(X^{\Omega'}, x^{\mu'})$ in every other allowable system of reference $(X^{\Omega'}, x^{\mu'})$. An *invariant property* of a 2-point field is a property held in common by all of its representations. Joint invariants, differential invariants, *etc.*, are defined as in the case of 1-point fields.

A *2-point tensor field* is a set of functions $T^{\mu\ldots\Omega\ldots}_{\nu\ldots\Theta\ldots}(X^\Omega, x^\mu)$ with a transformation law of the general form

$$\begin{aligned} T^{\mu'\ldots\Omega'\ldots}_{\nu'\ldots\Theta'\ldots} &= |(x'x)|^{-w}\,|(X'X)|^{-W}\big(\mathrm{sgn}\,(x'x)^y(\mathrm{sgn}\,(X'X)\big)^Y \times \\ &\times \frac{\partial x^{\mu'}}{\partial x^\mu}\cdots\frac{\partial X^{\Omega'}}{\partial X^\Omega}\cdots\frac{\partial x^\nu}{\partial x^{\nu'}}\cdots\frac{\partial X^\Theta}{\partial X^{\Theta'}}\,T^{\mu\ldots\Omega\ldots}_{\nu\ldots\Theta\ldots}. \end{aligned} \tag{8.1}$$

An *absolute 2-point tensor field* is one for which $w = W = y = Y = 0$. Names of 2-point fields with other values of the exponents w, W, y, and Y are assigned on a basis similar to the case of 1-point fields. 1-point tensor fields are special cases of 2-point tensor fields which are constant scalars with respect to transformations of one of the points. Thus, in all of the work preceding this section, the 1-point fields in space-time can be regarded as special cases of 2-point fields.

Let $\mathcal{S}_1$ be the 3-dimensional space $\mathcal{S}_{\mathrm{M}}$ of material points with group $\mathfrak{G}_{\mathrm{MA}}$ and let $\mathcal{S}_2$ be one of the 4-dimensional spaces $\mathcal{S}_{\mathrm{E}}$, $\mathcal{S}_{\mathrm{G}}$, or $\mathcal{S}_{\mathrm{L}}$ with group $\mathfrak{G}_{\mathrm{A}}$. The invariant theoretic problems of classical continuum mechanics can be phrased in terms of 2-point fields $F_\Delta(X^K, x^\mu)$. We call such a 2-point field a *world-material*

field. We have already met with an example of such a world-material field $(C^{-1})^{KL}(X^M, T)$ that we defined by setting

$$(C^{-1})^{KL}(X^M, T) = \overset{KL}{C^{-1}}(\overset{\mu}{x}(X^M, T)). \tag{8.2}$$

One readily verifies that $(C^{-1})^{KL}$ is a world-material tensor field having a transformation law of the general form (8.1). The dependence of this field on the space-time coordinates x^μ is somewhat special since it depends on the x^μ only in the combination $T = t(x^\mu)$. Now it is possible also to regard $(C^{-1})^{KL}$ as a 1-point field in space-time, $\overset{KL}{C^{-1}}(x^\mu)$. But these quantities do not have the convenient tensor law of transformation under the group $\mathfrak{G}_{\mathrm{MA}}$. Rather, the $\overset{KL}{C^{-1}}(x^\mu)$ have the odd transformation law

$$\overset{K'L'}{C^{-1}}(x^\mu) = \underset{K}{\overset{K'}{B}}(x^\mu)\, \underset{L}{\overset{L'}{B}}(x^\mu)\, \overset{KL}{C^{-1}}(x^\mu), \tag{8.3}$$

where the functions $\underset{K}{\overset{K'}{B}}$ are determined by

$$\underset{K}{\overset{K'}{B}}(x^\mu) = \frac{\partial \overset{K'}{Y}}{\partial X^K}\left(\overset{K}{X}(x^\mu)\right), \tag{8.4}$$

and the $\overset{K}{X}(x^\mu)$ are the functions representing the motion relative to the general coordinate system x^μ. Thus the transformation law of the $\overset{KL}{C^{-1}}(x^\mu)$ under the group $\mathfrak{G}_{\mathrm{MA}}$ depends on the motion $\overset{K}{X}(x^\mu)$ and cannot be written down without it. Of course, we used the motion to define the functions $(C^{-1})^{KL}(X^K, t)$, but they have a more convenient tensor law of transformation under $\mathfrak{G}_{\mathrm{MA}}$.

The concept of a 2-point field and a pair of spaces $\mathscr{S}_{\mathrm{E}}$, $\mathscr{S}_{\mathrm{G}}$, or $\mathscr{S}_{\mathrm{L}}$ and $\mathscr{S}_{\mathrm{M}}$ each with its own geometry leads naturally to the interpretation of a motion $X^K = \overset{K}{X}(x^\mu)$ as a mapping between the points of space-time and the material points of $\mathscr{S}_{\mathrm{M}}$. The mapping is one-to-one only in the direction $x^\mu \to X^K$. In the other direction $X^K \to x^\mu$, a single material point is mapped onto a 1-dimensional set of events, the world line of X^K.

The geometry of world-material fields is enriched still further by adding the *connectors* or *shifters* $g^\mu_K(X^M, x^\pi)$, $g^K_\mu(X^M, x^\pi)$ to the list of 1-point fields $g^{\mu\nu}$, t_π, $\Gamma^\varrho_{\sigma\pi}$, $\gamma^{\lambda\xi}$, $H^{KL\ldots}(X^M), \ldots, G_{KL}(X^M)$ which characterize the geometry of each of our two spaces separately. The connectors are 2-point world-material absolute tensor fields providing a linkage or connection between the two spaces. Quantities similar to the connectors we now introduce have been considered previously in [*19, 17*]. For our purposes here, we shall define the components of the connectors in a general system of coordinates (X^K, x^μ) as follows: Let there be a class of *preferred* Euclidean, Galilean, or Lorentz frames z^μ_0 and rectangular Cartesian material frames Z^K_0 such that the spatial frames z^i_0 and the material frames Z^K_0 are "at rest" relative to each other and whose axes coincide. With respect to such a system of reference (Z^K_0, z^μ_0) we assume that the connectors have the *joint canonical* form

$$g^\mu_K = (\delta^i_K, 0), \qquad g^K_\mu = (\delta^K_i, 0). \tag{8.5}$$

The components of the connectors in a general system of reference are then given by

$$g^{\mu}_{K}(X^M, x^\pi) = \delta^i_L \frac{\partial Z^L_0}{\partial X^K} \frac{\partial x^\mu}{\partial z^i_0}, \qquad g^K_\mu(X^M, x^\pi) = \delta^L_i \frac{\partial X^K}{\partial Z^L_0} \frac{\partial z^i_0}{\partial x^\mu}. \tag{8.6}$$

Stated more simply and directly, we assume that the connectors are absolute 2-point tensor fields such that by suitable choice of a Euclidean, Galilean, or Lorentz frame and a Cartesian material frame they are reducible to the joint canonical form (8.5). A pair of coordinate systems (Z^K_0, z^μ_0) in which we have (8.5) is called a *common frame.* If the components of the connectors corresponding to $\mathscr{S}_{\mathrm{E}}$ and $\mathscr{S}_{\mathrm{G}}$ are referred to an arbitrary Cartesian material frame Z^K and an arbitrary Euclidean or Galilean frame z^μ they will have the *canonical* form

$$\begin{aligned} g^\mu_K &= (S^i_K, 0), & g^K_\mu &= (S^K_i, V^K), \\ V^K &= -S^K_i V^i, & S^i_K S^K_j &= \delta^i_j, \end{aligned} \tag{8.7}$$

where V^i is the *relative velocity* of the origin of the spatial frame z^i and the material frame Z^K, and S^i_K is an orthogonal matrix representing the relative orientation of the two systems of axes Z^K and z^i. In Euclidean space-time, the S^K_i and V^i will be general functions of z^4, while in Galilean space-time they will be constants. Let V^μ be a world vector in $\mathscr{S}_{\mathrm{E}}$ or $\mathscr{S}_{\mathrm{G}}$ with the canonical form

$$V^\mu = (V^i, 1). \tag{8.8}$$

The Euclidean, Galilean and Lorentzian connectors satisfy the invariant relations

$$\begin{aligned} g^\mu_K g^L_\mu &= \delta^L_K, & g^\mu_K g^K_\nu &= \delta^\mu_\nu - V^\mu t_\nu, \\ g^K_\mu g^L_\nu g^{\mu\nu} &= G^{KL}, & G^{KL} g^\mu_K g^\nu_L &= g^{\mu\nu}, \\ g^\mu_K g^\nu_L \gamma_{\mu\nu} &= G_{KL}, & g^K_\mu g^L_\nu \gamma^{\mu\nu} &= G^{KL}. \end{aligned} \tag{8.9}$$

All of these relations can be verified by referring all quantities to the common frame. Since they are tensor equations holding in one frame, they will hold in a general system of coordinates.

As an illustration of the kind of world invariants of a motion in $\mathscr{S}_{\mathrm{E}}$ or $\mathscr{S}_{\mathrm{G}}$ that one can construct with the connectors, we consider the problem of defining a world tensor measure of the finite rotation of a motion relative to the common frame. The considerations given here are natural generalizations of those given in [*17*, § 4] to the case of arbitrary moving and deforming coordinate systems in space-time.

Consider the world tensor defined by

$$c_{\mu\nu} \equiv G_{KL}\big(\overset{M}{X}(x^\pi)\big)\, \partial_\mu X^K\, \partial_\nu X^L. \tag{8.10}$$

The canonical form of this tensor is

$$c_{\mu\nu} = \begin{vmatrix} c_{ij} & -c_{ik} v^k \\ -c_{jk} v^k & c_{kl} v^k v^l \end{vmatrix}, \qquad c_{ij} \equiv \delta_{KL}\, \partial_i Z^K\, \partial_j Z^L. \tag{8.11}$$

The quantities c_{ij} are the *spatial measures of finite deformation* introduced into elasticity theory by Cauchy and Green [*8*]. It follows immediately from the

canonical form (8.11) that det $c_{\mu\nu}=0$ so that $c_{\mu\nu}$ is a singular tensor. Its null eigenvector is the world velocity vector v^μ. Thus in defining a world spatial measure of finite deformation it proves more convenient to use the tensor $c^{*\mu\nu}$ defined by

$$c^{*\mu\nu} \equiv G^{KL}\,\partial_K \overset{\mu}{x}\,\partial_L \overset{\nu}{x}, \tag{8.12}$$

where $\overset{\mu}{x}(X^K, T)$ are the functions (4.5). The canonical form of $c^{*\mu\nu}$ is

$$c^{*\mu\nu} = \begin{bmatrix} (c^{-1})^{ij} & 0 \\ 0 & 0 \end{bmatrix}, \tag{8.13}$$

where the $(c^{-1})^{ij}$ are the components of the inverse of c_{ij}. $c^{*\mu\nu}$ has the null eigenvector t_μ and $(c^{-1})^{ij}$ is positive definite. Consider the eigenvalue equation

$$c^{*\mu\nu}\underset{\Omega}{n_\nu} = \underset{\Omega}{c}^{-1} g^{\mu\nu}\underset{\Omega}{n_\nu}, \qquad \Omega = 1, 2, 3. \tag{8.14}$$

This equation has 3 solutions $\left(\underset{\Omega}{n_\nu}, \underset{\Omega}{c}^{-1}\right)$ with positive $\underset{\Omega}{c}$ and vectors $\underset{\Omega}{n_\nu}$ satisfying the two conditions

$$g^{\mu\nu}\underset{\Omega}{n_\mu}\underset{\Theta}{n_\nu} = \delta_{\Omega\Theta}, \qquad v^\mu \underset{\Omega}{n_\mu} = 0. \tag{8.15}$$

The scalars $\left(\underset{\Omega}{c}^{\frac{1}{2}} - 1\right)$ are called the *principal extension ratios*. The canonical form of the vectors $\underset{\Omega}{n_\mu}$ is

$$\underset{\Omega}{n_\mu} = \left(\underset{\Omega}{n_i}, -v^i \underset{\Omega}{n_i}\right), \tag{8.16}$$

where the unit vectors $\underset{\Omega}{n_i}$ determine the *principal axes of strain in the deformed body* [8, 17]. All of these results follow from the canonical form of $c^{*\mu\nu}$, and we have simply placed them in world invariant form.

Now consider the world-material vector fields $\underset{\Omega}{N}(X^K, T)$ satisfying the eigenvalue equation

$$(C^{-1})^{KL}\underset{\Omega}{N_L} = \underset{\Omega}{C}^{-1} G^{KL}\underset{\Omega}{N_L}. \tag{8.17}$$

Since $(C^{-1})^{KL}$ is positive definite, the eigenvalues $\underset{\Omega}{C}^{-1}$ are all positive, and there exist three linearly independent eigenvectors $\underset{\Omega}{N_L}$ satisfying

$$G^{KL}\underset{\Omega}{N_K}\underset{\Theta}{N_L} = \delta_{\Omega\Theta}. \tag{8.18}$$

Consider next the world-material vector fields $\underset{\Omega}{n_K}(X^M, T)$ obtained from the fields $\underset{\Omega}{n_\mu}(x^\pi)$ according to the rule

$$\underset{\Omega}{n_K}(X^M, T) \equiv g^\mu_K\left(X^M, \overset{\pi}{x}\right)\underset{\Omega}{n_\mu}\left(\overset{\pi}{x}\right). \tag{8.19}$$

Since we have the identity $(8.9)_4$, it follows from (8.15) and (8.19) that

$$G^{KL}\underset{\Omega}{n_K}\underset{\Theta}{n_L} = \delta_{\Omega\Theta}. \tag{8.20}$$

Thus $\underset{\Omega}{N}_K$ and $\underset{\Omega}{n}_K$ are two sets of orthogonal unit vectors at (X^K, T). Therefore, there exists a unique matrix $R^K_L(X^M, T)$ satisfying

$$\underset{\Omega}{n}_L = R^K_L \underset{\Omega}{N}_K, \quad G^{MN} R^K_M R^L_M = G^{KL}, \quad R^K_L = \sum_{\Omega} \underset{\Omega}{n}_L \underset{\Omega}{N}_N G^{KN}. \tag{8.21}$$

When the material system of coordinates is rectangular Cartesian, R^K_L is an orthogonal matrix. The sense in which this matrix is a measure of the finite rotation of a motion and the relation in which it stands to the classical measure of infinitesimal rotation has been explained in [*19, 17*]. From the canonical forms of $c^{*\mu\nu}$ and $(C^{-1})^{KL}$ we see that c_{ij} and $(C^{-1})^{KL}$ have equal eigenvalues. Since these eigenvalues are absolute scalars under general transformations of the coordinates (X^K, x^μ), we shall have in general

$$\underset{\Omega}{c}\left(\overset{\mu}{x}\right) = \underset{\Omega}{C}^{-1}, \tag{8.22}$$

provided we order the two sets of eigenvalues appropriately. With this result we can show that the vector fields $\underset{\Omega}{n}_\mu$ given by

$$\underset{\Omega}{n}_\mu = \partial_\mu X^K \underset{\Omega}{N}_K\left(\overset{K}{X}, t\right) \sqrt{\underset{\Omega}{C}} \tag{8.23}$$

satisfy all of the equations (8.14) and (8.15). If the eigenvalues of C_{KL} are distinct so are the eigenvalues of $c^{*\mu\nu}$ and the eigenvectors $\underset{\Omega}{N}_K$ and $\underset{\Omega}{n}_\mu$ are uniquely determined. This is not true if two or more of the eigenvalues are equal. However, in the case of distinct eigenvalues, the motion determines a unique matrix R^K_L provided we order each set of eigenvectors in some definite way such as that corresponding to the equalities (8.22), (8.23).

Multiplying (8.23) by $g^\mu_M \underset{\Omega}{N}_L$ and summing on μ and Ω we get

$$\begin{aligned} G_{MK} R^K_L &= \sum_{\Omega} \underset{\Omega}{n}_\mu \underset{\Omega}{N}_L g^\mu_M = \sum_{\Omega} \underset{\Omega}{n}_M \underset{\Omega}{N}_L = \partial_\mu X^K g^\mu_M \sum_{\Omega} \sqrt{\underset{\Omega}{C}} \underset{\Omega}{N}_K \underset{\Omega}{N}_L, \\ R_{ML} &= \partial_\mu X^K g^\mu_M (C^{\frac{1}{2}})_{KL}. \end{aligned} \tag{8.24}$$

Multiplying this last equation through by $(C^{-\frac{1}{2}})^{LN} g^M_\nu$ and using $(8.9)_2$ we get finally

$$\partial_\nu X^K = t_\nu V^\mu \partial_\mu X^K + (C^{-\frac{1}{2}})^{KL} R_{ML} g^M_\nu. \tag{8.25}$$

The canonical form of this world-material tensor equation is

$$\partial_i X^K = (C^{-\frac{1}{2}})^{KL} R_{ML} S^M_i, \tag{8.26}$$

$$\frac{\partial X^K}{\partial T} = (C^{-\frac{1}{2}})^{KL} R_{ML} V^M + V^i \partial_i X^K + \frac{\partial X^K}{\partial T}. \tag{8.27}$$

Equation (8.26) corresponds to the result (4.19) of [*17*]. The last equation is an identity satisfied as a consequence of (8.26) and $(8.7)_2$. Equation (8.26) is the familiar decomposition of a deformation $\partial_i X^K$ into a pure stretching without rotation followed by a rigid rotation. In Euclidean space-time, the orthogonal matrix S^M_i will in general depend on the time. As we have said, it represents the time dependent relation between the material coordinate axes and the moving, rotating axes of the Euclidean frame.

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Die Stabilität der Strömung in einem gekrümmten Kanal

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Inhalt

1. Einleitung

Seit G. I. Taylor im Jahre 1923 seine Untersuchungen über die Stabilität der Strömung einer zähen Flüssigkeit veröffentlichte [*1*], die sich zwischen zwei rotierenden achsengleichen Zylindern befindet, gewann die Betrachtung der dort zum ersten Male aufgetretenen stehenden Wirbel mehr und mehr an Interesse.

Das Wesentliche an dem Taylorschen Modell ist es, daß eine Strömung längs einer gekrümmten Wand vorhanden ist, die sich in einem labilen Zustand befindet. W. R. Dean [*2*] übertrug im Jahr 1928 den Taylorschen Grundgedanken der Störung einer laminaren Strömung durch Überlagerung einer stehenden wirbelartigen Instabilität auf die Strömung in einem gekrümmten, unendlich hohen Kanal; die Wände des Kanals werden dabei von zwei achsengleichen Zylindern gebildet. Kürzlich wurde dieselbe Aufgabenstellung von C. S. Yih und W. M. Sangster [*3*] aufgegriffen, wobei diese Autoren die Arbeit [*2*] offenbar nicht kannten.

Der Ansatz einer Taylorschen Instabilität in die vollen Navier-Stokesschen Gleichungen der rotationssymmetrischen Kanalströmung führt auf ein Eigenwertproblem der Art, wie es schon bei Taylor auftritt. Als Parameter treten darin die Wirbeldicke der angesetzten Instabilität und deren Anfachungskonstante auf, während wir als den davon abhängigen Eigenwertparameter eine Größe $\mathrm{Re}^2 \frac{d}{R_1}$ betrachten wollen (d = Kanalbreite, R_1 = Krümmungsradius der inneren Kanalberandung und Re = Reynoldssche Zahl der Grundströmung, gebildet mit d).

Unter den Eigenwerten interessieren dabei vor allem diejenigen neutraler Wirbel und unter ihnen wiederum derjenige kleinsten Betrages, den man gewöhnlich als „kritischen Wert" bezeichnet. Seine Bedeutung liegt darin, daß eine

Strömung sicher dann nicht gegen Instabilitäten der betrachteten Art anfällig ist, wenn der ihr zukommende Wert des Parameters $\mathrm{Re}^2 \frac{d}{R_1}$ unter dem kritischen liegt. Zu diesem kritischen Wert gehört auch eine bestimmte Wirbeldicke, die also den gefährlichsten der neutralen Wirbel eigentümlich ist. Das Aussehen eines solchen Wirbels kann man beschreiben, wenn man die zu dem kleinsten Eigenwert gehörenden Eigenfunktionen kennt, die gerade die Geschwindigkeitskomponenten der Wirbel darstellen.

In der Arbeit von W. R. DEAN wird der kritische Wert angegeben, und es wird gezeigt, daß auch gedämpfte und angefachte Störungen existieren. Über die Eigenfunktionen und damit das Aussehen der Wirbel werden keine Aussagen gemacht. YIH und SANGSTER behandeln nur neutrale Störungen und geben einen kritischen Wert an, der zu dem Deanschen im Verhältnis 1:150 steht.

Die vorliegende Arbeit verfolgt zwei Ziele: Einmal wird eine Neubehandlung des Eigenwertproblems für neutrale Störungen mitgeteilt, die es ermöglicht, die Eigenwerte kleinsten Betrages in Abhängigkeit von der Dicke der angesetzten Wirbel genau zu bestimmen. Es wird sich zeigen, daß die so gefundenen Werte mit den von DEAN angegebenen übereinstimmen. Des weiteren werden wir auch die Eigenfunktionen finden, die zu diesen Eigenwerten gehören, so daß damit die Gestalt der auftretenden Wirbel beschrieben werden kann. Die Genauigkeit unseres Vorgehens ermöglicht es, die Resultate numerisch zu sichern.

2. Die Störungsdifferentialgleichungen

a) Zur Untersuchung der Strömung in einem gekrümmten Kanal hat man von den vollen Navier-Stokesschen Bewegungsgleichungen dieser Strömung auszugehen. Wir wählen die übliche Darstellung in Zylinderkoordinaten; dabei seien r die radiale Richtung, φ das Winkelmaß und z die Koordinate in Richtung der Zylinderachse. R_1 sei der Radius des inneren, R_2 der des äußeren begrenzenden Zylinders (Fig. 1). Die Geschwindigkeitskomponenten werden folgendermaßen benannt:

φ-Richtung: u

r-Richtung: v

z-Richtung: w.

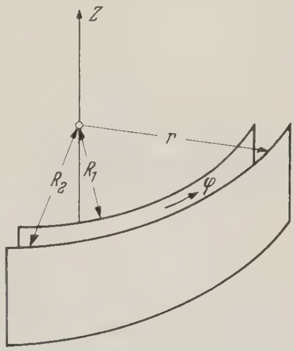

Fig. 1. Das Koordinatensystem

Man kennt eine exakte, stationäre Lösung der infolge der Rotationssymmetrie des Problems vereinfachten Bewegungsgleichungen, die die Randbedingungen $u(R_1) = v(R_1) = w(R_1) = 0$, $u(R_2) = v(R_2) = w(R_2) = 0$ befriedigt:

$$u_0 = \frac{R_2}{2\mu} \frac{\partial p_0}{\partial \varphi} \left\{ \frac{R_2^2 \ln(R_1/R_2)}{R_2^2 - R_1^2} \left(\frac{r}{R_2} - \frac{R_2}{r} \right) + \frac{r}{R_2} \ln \frac{r}{R_2} \right\},$$

$$v_0 = 0,$$

$$w_0 = 0 \qquad \text{(GOLDSTEIN [4], S. 315).}$$

Für den Druck p_0 ergibt sich dabei

$$p_0 = k\varphi + f(r),$$

also

$$\frac{\partial p_0}{\partial \varphi} = k, \quad \text{konstant.}$$

b) Um die Stabilität der Grundströmung u_0, v_0, w_0 zu untersuchen, überlagern wir dieser kleine Störungen u^*, v^*, w^*; d.h. wir gehen mit dem Ansatz $u = u_0(r) + u^*(r, z, t)$; $v = v^*(r, z, t)$; $w = w^*(r, z, t)$; $p = p_0(r, \varphi) + p^*(r, z, t)$ in die vollen Navier-Stokesschen Differentialgleichungen. Diese werden unter Berücksichtigung der Tatsache, daß u_0 und p_0 stationäre Lösungen sind, hinsichtlich der gesternten Störgeschwindigkeiten linearisiert. Im Hinblick auf die geometrischen Verhältnisse unserer Aufgabe machen wir dann Gebrauch von der Annahme, daß die Breite des durchströmten Kanals klein sei gegenüber den Radien seiner Wände, also $R_2 - R_1 = d \ll R_1$. Da die Störungen g^* naturgemäß auf das Innere des Kanals beschränkt sind, folgt in bekannter Weise $\frac{1}{r^2} g^* \ll \frac{1}{r} g_r^*$ und $\frac{1}{r} g_r^* \ll g_{rr}^*$.

Mit diesen Annahmen, die eine erste Approximation hinsichtlich der geometrischen Verhältnisse darstellen, reduzieren sich die Störungsdifferentialgleichungen schließlich auf

$$u_t^* + v^*\left(\frac{du_0}{dr} + \frac{1}{r} u_0\right) = \nu(u_{rr}^* + u_{zz}^*), \tag{1.1}$$

$$v_t^* - \frac{2}{r} u^* u_0 = -\frac{1}{\varrho} p_r^* + \nu(v_{rr}^* + v_{zz}^*), \tag{1.2}$$

$$w_t^* = -\frac{1}{\varrho} p_z^* + \nu(w_{rr}^* + w_{zz}^*), \tag{1.3}$$

$$v_r^* + w_z^* = 0. \tag{1.4}$$

Als Randbedingungen hat man das Verschwinden der Störgeschwindigkeiten an den Berandungen zu fordern, also

$$u^*(R_1) = v^*(R_1) = w^*(R_1) = 0$$
$$u^*(R_2) = v^*(R_2) = w^*(R_2) = 0.$$

Für das Folgende führen wir anstelle von r eine neue Koordinate η ein:

$$\eta = \frac{R_2 - r}{R_2 - R_1} \qquad \begin{array}{l} r = R_2 \to \eta = 0 \\ r = R_1 \to \eta = 1. \end{array}$$

Für die gesternten Störglieder machen wir nun nach dem Vorbild von TAYLOR den speziellen Ansatz

$$\begin{array}{ll} u^* = u_1(\eta) \cos\alpha z\, e^{\beta t} & p^* = p_1(\eta) \cos\alpha z\, e^{\beta t} \\ v^* = v_1(\eta) \cos\alpha z\, e^{\beta t} & \alpha = 2\pi/\lambda \\ w^* = w_1(\eta) \sin\alpha z\, e^{\beta t} & \lambda/2 = \text{Wirbeldicke.} \end{array}$$

Unter der schon oben benutzten Annahme $d \ll R_1$ wird die Grundströmung $u_0(r)$ in demselben Grad der Approximation, in dem die Gleichungen (1) gelten, zu

$$u_0(\eta) = K \frac{d^2}{R_1^2} (\eta^2 - \eta),$$

wobei der konstante Wert $\frac{R_2}{2\mu} \frac{\partial p_0}{\partial \varphi} = K < 0$ gesetzt ist.

Um die Störungsgleichungen dimensionslos zu machen, wählen wir als Bezugsgeschwindigkeit

$$\bar{u}_0 = \int_0^1 u_0(\eta)\, d\eta = -\frac{1}{6} K \frac{d^2}{R_1^2};$$

das führt auf die Reynoldssche Zahl $\mathrm{Re} = \frac{\bar{u}_0 d}{\nu}$, und wir schreiben

$$U(\eta) = \frac{1}{6} \frac{u_0}{\bar{u}_0} = \eta - \eta^2.$$

Außerdem seien

$$\sigma = \alpha d, \quad \tau^2 = \sigma^2 + \frac{\beta d^2}{\nu}, \quad S = 72\, \mathrm{Re}^2 \frac{d}{R_1}, \quad {}' = \frac{d}{d\eta}, \quad u = -u_1, \quad v = 6\, \mathrm{Re}\, v_1.$$

Nach Elimination des Stördrucks p_1 und der Geschwindigkeitskomponenten w_1 führt dies auf die Gleichungen

$$L_1 u = U' v \qquad \begin{array}{l} u(0) = v(0) = v'(0) = 0 \\ u(1) = v(1) = v'(1) = 0. \end{array} \tag{2.1}$$

$$L_2 L_1 v = -S \sigma^2 U u, \tag{2.2}$$

mit

$$L_1 = \frac{d^2}{d\eta^2} - \tau^2 \quad \text{und} \quad L_2 = \frac{d^2}{d\eta^2} - \sigma^2.$$

Die Randbedingungen für v' ergeben sich aus denen für w nach der Gleichung

$$v' - \sigma w = 0, \tag{2.3}$$

die aus der Kontinuitätsgleichung (1.4) fließt.

Damit ist unser Problem in diejenige Form gebracht, an die sich die weitere mathematische Behandlung anschließen soll.

Die Gleichungen (2) finden sich im wesentlichen schon bei Dean [2], und Yih und Sangster behandeln ebenfalls dieses System für den neutralen Fall $\tau = \sigma$. Dort wird die Variable $\xi = 1 - \eta$ eingeführt, und einige Bezeichnungen sind anders gewählt. Ein Zusammenhang mit dem früher behandelten Problem der Stabilität einer Grenzschichtströmung längs einer konkaven Wand ([5], [6]) läßt sich nun leicht herstellen: Die Störungsdifferentialgleichungen sind hier dieselben wie dort, jedoch haben sich die Randbedingungen insofern geändert, als an die Stelle des äußeren Randes der Grenzschicht ($\eta = \infty$) nun der innere Zylinder ($\eta = 1$) tritt, und entsprechend hat man hier natürlich ein anderes Grundprofil $U(\eta)$.

3. Die Vorbereitung des Eigenwertproblems

Die Gleichungen (2.1) und (2.2) stellen ein Eigenwertproblem dar, dessen Parameter σ, τ und S sind. Diejenigen Wirbel, die den Grenzzustand zwischen gedämpften und angefachten Störungen darstellen, bezeichnen wir als neutrale Störungen; sie sind durch $\beta = 0$ $(\tau = \sigma)$ gekennzeichnet. Sie interessieren besonders, da sie die kritische Kurve des Parameters S liefern, wenn wir von der üblichen Auffassung ausgehen, daß zu einer vorgegebenen Wirbeldicke (beschrieben durch σ) der zugehörige Wert S zu berechnen ist. Wir werden uns im folgenden auf diesen neutralen Fall beschränken.

Für $\tau = \sigma$ lauten die Gleichungen (2)

$$(3.1) \qquad L u = (1 - 2\eta) v \qquad u(0) = v(0) = v'(0) = 0,$$

$$(3.2) \qquad L^2 v = - S\sigma^2 (\eta - \eta^2) u \qquad u(1) = v(1) = v'(1) = 0,$$

$$L = \frac{d^2}{d\eta^2} - \sigma^2.$$

Die Aufgabe besteht darin, den kleinsten Eigenwert S in Abhängigkeit von σ zu finden. Dazu ist zu bemerken: Führen wir an Stelle von η die Koordinate $\bar{\eta} = 1 - \eta$ ein, so gelangen wir zu dem System $(\bar{u} = u,\ \bar{v} = -v)$

$$L\bar{u} = (1 - 2\bar{\eta})\bar{v}$$

$$L^2\bar{v} = S\sigma^2(\bar{\eta} - \bar{\eta}^2)\bar{u}$$

mit den Randbedingungen

$$\bar{u}(0) = \bar{v}(0) = \bar{v}'(0) = 0$$

$$\bar{u}(1) = \bar{v}(1) = \bar{v}'(1) = 0.$$

Dieses System unterscheidet sich von (3) lediglich durch das Vorzeichen von S. Hat also (3) für den Eigenwert $S = +A$ eine Lösung, so hat das transformierte System eine Lösung für $S = -A$, in anderen Worten: Mit $S = +A$ ist stets auch $S = -A$ ein Eigenwert, oder es treten stets zwei Eigenwerte gleichen Betrags auf, die sich nur durch das Vorzeichen unterscheiden. Die Ursache dieser Erscheinung ist in der Willkür zu suchen, die in der Wahl der Variablen η bzw. $\bar{\eta}$ liegt. Es wird sich zeigen, daß die Eigenwerte reell sind, und für unser Problem ist immer der positive maßgebend. Jedoch hat das Auftreten eines zweiten Eigenwerts gleichen Betrags eine noch zu erläuternde Bedeutung für die Berechnung der unteren Schranken und für das Iterationsverfahren, welches angewendet werden soll.

Zur weiteren Behandlung verwandeln wir das System (3) in ein System von zwei Integralgleichungen. Das geschieht mit Hilfe der bereits von H. WITTING [7] berechneten Greenschen Funktionen $M(\eta, \xi)$ und $N(\eta, \xi)$ der linken Seiten von (3). Sie haben die Gestalt

$$M(\eta, \xi) = -\frac{1}{\sigma \sinh \sigma} \sinh \sigma\eta \sinh \sigma(\xi - 1) \qquad \text{in } \eta \leqq \xi$$

und $M(\xi, \eta) = M(\eta, \xi)$,

$$N(\eta, \xi) = \{[N_1(\sigma) + N_2(\sigma)]\, f_2(\sigma(\xi - 1)) - \sigma N_1(\sigma)\, f_1(\sigma(\xi - 1))\}\, f_2(\sigma\eta) +$$
$$+ \{[N_1(\sigma) - N_2(\sigma)]\, f_1(\sigma(\xi - 1)) + \sigma N_1(\sigma)\, f_2(\sigma(\xi - 1))\}\, f_1(\sigma\eta) \qquad \text{für } \eta \leqq \xi$$

und $N(\eta,\xi) = N(\xi,\eta)$. Dabei sind

$$N_1(\sigma) = \frac{1}{2\sigma}\,\frac{\sinh\sigma}{\sigma^2\sinh^2\sigma - \sigma^4} \qquad f_1(y) = y\sinh y$$

$$N_2(\sigma) = \frac{1}{2}\,\frac{\cosh\sigma}{\sigma^2\sinh^2\sigma - \sigma^4} \qquad f_2(y) = \sinh y - y\cosh y.$$

Mit Hilfe dieser Greenschen Funktionen entsteht aus (3) das System

(4.1) $$u(\eta) = -\int_0^1 M(\eta,\xi)\,(1-2\xi)\,v(\xi)\,d\xi,$$

(4.2) $$v(\eta) = S\sigma^2\int_0^1 N(\eta,\xi)\,(\xi-\xi^2)\,u(\xi)\,d\xi.$$

Das System (4) erlaubt einige Abschätzungen für $S(\sigma^2)$, die der tatsächlichen Berechnung vorausgehen sollen.

4. Untere Schranken für die kritische Kurve

Aus den Integralgleichungen (4) läßt sich durch Einsetzen von (4.1) in (4.2) eine einzige Gleichung herstellen:

(5.1) $$v(\eta) = S\int_0^1 K(\eta,t)\,v(t)\,dt.$$

Dabei ist der Kern $K(\eta,t)$ dieser Gleichung von der Form

(5.2) $$K(\eta,t) = -(1-2t)\int_0^1 \sigma^2 N(\eta,\xi)\,(\xi-\xi^2)\,M(\xi,t)\,d\xi.$$

Der Kern $K(\eta,t)$ ist stetig und nicht symmetrisch.

Nach J. Schur (Hellinger-Toeplitz [8], S. 1550) genügen die Eigenwerte λ_i einer Integralgleichung

$$f(x) = \lambda\int_0^1 K(x,y)\,f(y)\,dy$$

mit beliebigem quadratisch integrablem Kern $K(x,y)$ der Beziehung

$$\sum_1^\infty \frac{1}{|\lambda_i|^2} \le \int_0^1\!\!\int_0^1 |K(x,y)|^2\,dx\,dy;$$

dabei tritt jeder Eigenwert so oft auf, wie seine Vielfachheit angibt. Entsprechend gilt für die iterierten Kerne

(6) $$\sum_1^\infty \frac{1}{|\lambda_i|^{2m}} \le \int_0^1\!\!\int_0^1 |K^{(m)}(x,y)|^2\,dx\,dy.$$

Diese Beziehungen liefern die in unserem Falle besonders wünschenswerten unteren Schranken für den kleinsten Eigenwert der Gleichung (5).

Im 3. Abschnitt wurde gezeigt, daß je zwei Eigenwerte gleichen Betrags auftreten. Damit ist $S_2 = -S_1$, und die aus (6) folgende Abschätzung für S_1 verschärft sich zu

$$\frac{2}{|S_1|^{2m}} \le \int_0^1\!\!\int_0^1 |K^{(m)}(\eta,t)|^2\,d\eta\,dt.$$

In dieser Abschätzung bereitet die Berechnung der iterierten Kerne gewisse numerische Schwierigkeiten. Zunächst ist nach (5.2) der Kern $K(\eta, t) = K^{(1)}(\eta, t)$ zu gewinnen; diese Integration über ξ wurde mit den Schrittweiten $h = 0{,}2$ und $h = 0{,}1$ durchgeführt. Daraus ergibt sich, wiederum durch Anwenden der auf Gebiete übertragenen Trapezregel, $\int_0^1\int_0^1 K^2\,d\eta\,dt$, und damit die Abschätzung

$$|S_1| \geqq E_1 = \frac{\sqrt{2}}{\left\{\int_0^1\int_0^1 K^2(\eta, t)\,d\eta\,dt\right\}^{\frac{1}{2}}}.$$

Da wir numerisch integrieren, ist die Angabe des Doppelintegrals mit einem gewissen Fehler behaftet. Eine überschlägige Fehlerbetrachtung zeigt, daß die Angabe von $E_1(\sigma)$ in Fig. 2 (gewonnen aus den Werten für $h = 0{,}2$ und $h = 0{,}1$ durch Extrapolation mit der bekannten überschlägigen Überlegung) sicher auf 1% genau ist.

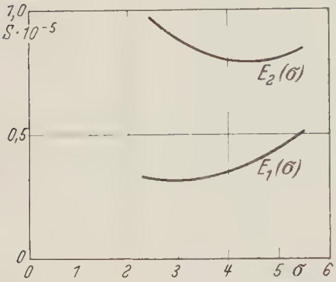

Fig. 2. Untere Schranken für $|S_1(\sigma)|$. $E_1(\sigma) < E_2(\sigma) \leqq |S_1(\sigma)|$

Ebenso ergibt sich mit dem iterierten Kern

$$K^{(2)}(\eta, t) = \int_0^1 K(\eta, u)\,K(u, t)\,du$$

die Abschätzung

$$S_1 \geqq E_2 = \frac{2^{\frac{1}{4}}}{\left\{\int_0^1\int_0^1 K^{(2)^2}(\eta, t)\,d\eta\,dt\right\}^{\frac{1}{4}}}.$$

Die in Fig. 2 angegebenen Werte für E_2 sind auf 3% genau.

Eine weitere Verbesserung dieser Abschätzungen mit Hilfe der höheren iterierten Kerne ist möglich. Dabei ist jedoch zu bedenken, daß der Integrationsfehler die Angabe höherer iterierter Kerne in steigendem Maße unsicher macht und daß außerdem die zu erwartende Verbesserung der unteren Schranke von Schritt zu Schritt geringer wird. Wir begnügen uns deshalb mit der Angabe der Schranke $E_2(\sigma)$ — die schon sehr brauchbar ist — und gehen zur Berechnung der Eigenwerte über. An und für sich macht das genaue Verfahren, das im folgenden verwendet wird, die Angabe unterer Schranken überflüssig. Sie sollen in diesem Fall auch nichts mehr bringen als eine vorläufige Aussage über die Größe des zu erwartenden Eigenwertes S_1.

5. Berechnung der Eigenwerte kleinsten Betrags

Die Berechnung des kleinsten Eigenwerts S_1 soll ihren Ausgang vom System (7) nehmen. Wir verwenden dazu ein Iterationsverfahren, das bereits in [*6*] und später in [*7*] erfolgreich durchgeführt wurde. Die exakte Fundierung des Verfahrens findet sich bei H. WIELANDT [*9*].

Dabei gehen wir von der Anfangsfunktion $v^{(0)}(\xi) = 16(\xi^2 - \xi)^2$ aus, die die Randbedingungen $v(0) = v'(0) = 0$ und $v(1) = v'(1) = 0$ erfüllt, und berechnen die weiteren Iterierten nach der Vorschrift

$$u^{(\nu)}(\eta) = -\int_0^1 M(\eta, \xi)\,(1 - 2\xi)\,v^{(\nu)}(\xi)\,d\xi, \tag{7.1}$$

$$v^{(\nu+1)}(\eta) = \int_0^1 \sigma^2 N(\eta, \xi)\,(\xi - \xi^2)\,u^{(\nu)}(\xi)\,d\xi. \tag{7.2}$$

In der Arbeit [6] waren die iterierten Funktionen durch elementare Integration ermittelt worden; das wäre auch hier möglich, jedoch ziehen wir jetzt der Einfachheit halber eine numerische Integration vor. Dazu wird durchgängig die Trapezregel verwendet, die in (7.1) mindestens teilweise unumgänglich ist, da $M(\eta, \xi)$ als Greensche Funktion einer Differentialgleichung 2. Ordnung einen Sprung in der ersten Ableitung besitzt. Aus Gründen der einheitlichen rechnerischen Durchführung wird auch in (7.2) mit ihr gearbeitet.

Während eine Iteration durch elementares Integrieren den exakten kleinsten Eigenwert mit beliebiger Genauigkeit liefert, dürfen wir das jedoch hier nicht mehr erwarten. Die Ersetzung der Integrale durch endliche Summen bedeutet ja eine Ersetzung des Systems (4) durch ein System linearer Gleichungen, das dann iterativ gelöst wird. Durch die Iteration wird nun die Lösung dieses letzteren Systems beliebig genau geliefert, und es hängt von der gewählten Schrittweite ab, wieweit diese Lösung mit der des Systems (4) bzw. (7) übereinstimmt.

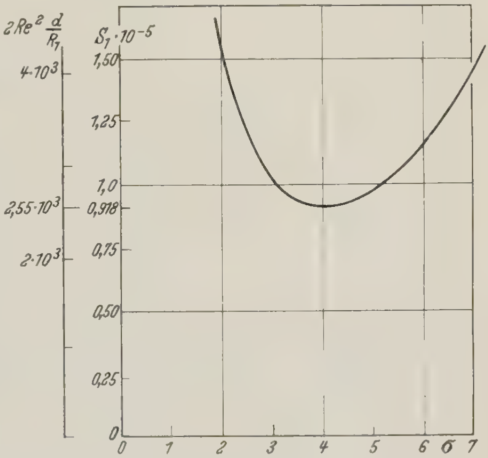

Fig. 3. Kleinster Eigenwert $S_1(\sigma)$; kritische Kurve: $2\,\mathrm{Re}^2 \frac{d}{R_1} = \frac{1}{36} S_1$

Wegen des Auftretens von je zwei Eigenwerten gleichen Betrags haben wir mit dem Eintreten des Oszillationsfalls der Iteration zu rechnen ([9], S. 126). Dann approximieren zwar die Iterierten noch nicht die gesuchten Eigenfunktionen, aber diese stellen sich als Linearkombination zweier aufeinanderfolgender Iterierter dar. Betrachtet man etwa die Iterierten $u^{(n)}$ und $u^{(n+1)}$, so werden die zum kleinsten Eigenwert S_1 gehörenden Eigenfunktionen durch

$$u \approx u^{(n)} + S_1 u^{(n+1)} \tag{8.1}$$

und

$$v \approx v^{(n)} + S_1 v^{(n+1)} \tag{8.2}$$

approximiert.

Aus demselben Grund wird hier nicht der Quotient $\frac{\mathfrak{G} f^{(\nu)}}{\mathfrak{G} f^{(\nu+1)}}$ linearer Funktionale gegen S_1 konvergieren, sondern es geht $\frac{\mathfrak{G} f^{(\nu)}}{\mathfrak{G} f^{(\nu+2)}} \to S_1^2$.

Die Rechnung wurde mit den Schrittweiten $h = 0{,}2$; $h = 0{,}1$ und $h = 0{,}05$ für $\sigma = 1, 2, 3, 3.4, 4, 5, 6, 7$ durchgeführt. Die in Fig. 3 angegebene Kurve $S_1(\sigma)$ ergibt sich daraus durch quadratische Extrapolation der sehr schnell konvergierenden Werte für $h \to 0$.

6. Die Eigenfunktionen und das Aussehen der gestörten Strömung

Wie oben schon erwähnt wurde, liefert das Iterationsverfahren nicht nur den kleinsten Eigenwert, sondern auch die zugehörigen Eigenfunktionen. Die Kenntnis dieser Eigenfunktionen gestattet es, das Aussehen der entstehenden Wirbel

zu beschreiben. Die gefährlichsten unter den neutralen Wirbeln sind diejenigen, zu deren Wirbeldicke der kleinste der im 5. Abschnitt errechneten Eigenwerte gehört. Das Minimum der Kurve $S_1(\sigma)$ liegt bei $\sigma = 4$. $\sigma = 4$ bedeutet, daß die Dicke eines solchen Wirbels $\frac{\lambda}{2} = \frac{\pi}{4} d = 0{,}79 d$ ist. In Fig. 4 sind die Eigenfunktionen für $\sigma = 4$ angegeben. $u(\eta)$ und $v(\eta)$ ergeben sich nach (8), während sich $w(\eta)$ aus der Beziehung (2.3)

$$w(\eta) = \frac{1}{\sigma} v'(\eta)$$

errechnet.

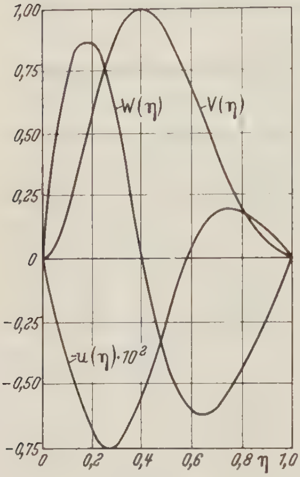

Fig. 4. Die Eigenfunktionen für die kritische Wirbeldicke $\lambda/2 = 0{,}79\, d$ ($\sigma = 4$). $\eta = 0$ entspricht $r = R_2$ (äußerer Zylinder), $\eta = 1$ entspricht $r = R_1$ (innerer Zylinder)

Die Komponenten der gestörten Strömung sind

$$u_0(\eta) + u_1(\eta) \cos \alpha z, \quad v_1(\eta) \cos \alpha z \quad \text{und} \quad w_1(\eta) \sin \alpha z .$$

Über die tatsächliche Größe der Wirbelkomponenten kann nichts gesagt werden, da die Eigenlösungen u, v, w, die den Störgeschwindigkeitsamplituden u_1, v_1, w_1 proportional sind, als Lösungen eines homogenen Systems nur bis auf einen Faktor zu bestimmen sind; auch die relativen Größenverhältnisse von u, v und w entsprechen nicht denen von u_1, v_1 und w_1, da diese in der dimensionslosen Darstellung mit verschiedenen Faktoren versehen wurden.

Dennoch erlauben es uns die Eigenfunktionen, die Gestalt der entstehenden Wirbel anzugeben. Fig. 5a zeigt einen Querschnitt senkrecht zu den Wirbelachsen, wobei der Betrachter in Richtung der Strömung blickt. Fig. 5b zeigt die Profilformen in Strömungsrichtung, also $\tilde{u}(\eta) = u_0(\eta) + u_1(\eta) \cos \alpha z$. Die ebenen Schnitte senkrecht zu der Achse der Kanalberandungen sind dabei in den folgenden Höhen gelegt:

z	$\tilde{u} = u_0 + u_1 \cos \frac{2\pi}{\lambda} z$	$\tilde{v} = v_1 \cos \frac{2\pi}{\lambda} z$	$\tilde{w} = w_1 \sin \frac{2\pi}{\lambda} z$
$k\lambda$	$u_0 + u_1$	v_1	0
$\frac{4k+1}{4}\lambda$	u_0	0	w_1
$\frac{4k+2}{4}\lambda$	$u_0 - u_1$	$-v_1$	0
$\frac{4k+3}{4}\lambda$	u_0	0	$-w_1$

$$k = 0, \ \pm 1, \ \pm 2, \ldots .$$

Zur Zeichnung der Profile $\tilde{u}(\eta)$ ist angenommen, daß $\operatorname{Max} |u_1(\eta)| = \frac{1}{10} \operatorname{Max} |u_0(\eta)|$ sei. Die angegebenen Wirbel sind diejenigen der Dicke $\lambda/2 = 0{,}79 d$.

Die Kerne dieser Wirbel sind etwas nach der äußeren Kanalwand hin verschoben; diese Erscheinung rührt von der stabilisierenden Wirkung der konvexen

inneren Kanalberandung her. Die Grundprofile werden abwechselnd nach innen und nach außen ausgebeult, während die Störkomponente u_1 in der Höhe der Wirbelkerne verschwindet.

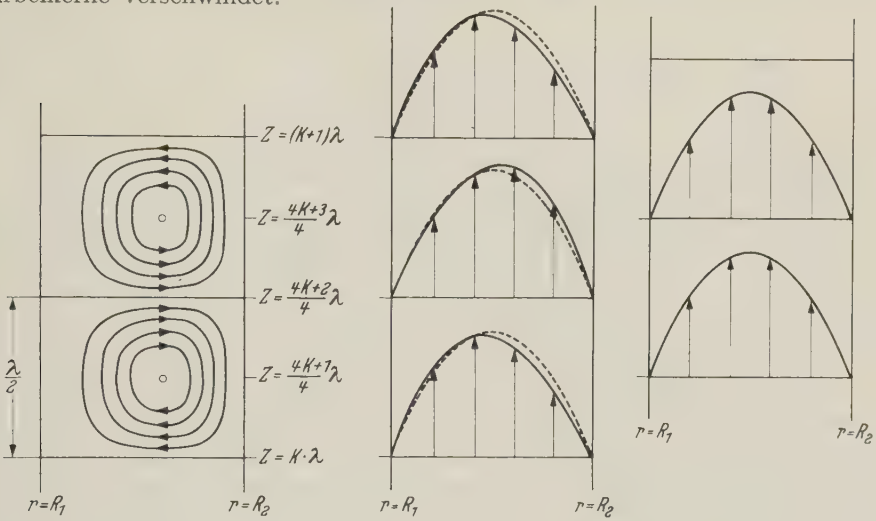

Fig. 5a u. b. Das Aussehen neutraler Wirbel. a Schnitt senkrecht zu den Wirbelachsen; b Strömungsprofile in Strömungsrichtung (φ-Richtung)

7. Vergleichende Betrachtungen

Es besteht die Notwendigkeit, noch ein Wort zu den numerischen Ergebnissen im 5. Abschnitt zu sagen. Der Eigenwert $S(\sigma)$ hatte die Bedeutung

$$S = 72\left(\frac{\bar{u}_0 d}{\nu}\right)^2 \frac{d}{R_1}.$$

Bei den Taylorschen rotierenden Zylindern erscheint ein entsprechender Parameter $T = 2\,\mathrm{Re}^2 \frac{d}{R_1}$, und bei der Untersuchung von Grenzschichten längs konkaven Wänden tritt $\mu = 2\,\mathrm{Re}^2 \frac{\delta}{R}$ auf. Die Reynoldsschen Zahlen sind dabei jeweils auf die maximale Grundgeschwindigkeit und auf Spaltbreite d bzw. Grenzschichtdicke δ bezogen. Wegen $|\bar{u}_0| = \frac{2}{3}\,\mathrm{Max}\,|u_0| = \frac{2}{3}\,u_M$ ist

$$2\left(\frac{u_M d}{\nu}\right)^2 \frac{d}{R_1} = \frac{1}{16}\,S.$$

Mit diesem Wert wird ein Vergleich möglich. In der Umgebung des Minimums hatten sich folgende Werte ergeben (die angegebenen Ziffern sind als sicher zu betrachten):

σ	$S_1 = 72\left(\frac{\bar{u}_0 d}{\nu}\right)^2 \frac{d}{R_1}$	$2\left(\frac{\bar{u}_0 d}{\nu}\right)^2 \frac{d}{R_1}$	$2\left(\frac{u_M d}{\nu}\right)^2 \frac{d}{R_1}$
3	$1{,}014 \cdot 10^5$	$2{,}82 \cdot 10^3$	$6{,}34 \cdot 10^3$
4	$0{,}918 \cdot 10^5$	$2{,}55 \cdot 10^3$	$5{,}74 \cdot 10^3$
5	$0{,}990 \cdot 10^5$	$2{,}75 \cdot 10^3$	$6{,}19 \cdot 10^3$

Das Minimum von $S_1(\sigma)$ liegt bei $\sigma = 4$ und hat den Wert

$$S_{1\,\mathrm{Min}} = 91\,800 \quad \text{oder} \quad 2\left(\frac{u_M d}{\nu}\right)^2 \frac{d}{R_1} = 5740\,.$$

W. R. DEAN [2] gibt hierfür den Wert $N = 36\left(\frac{a}{d}\right)^{\frac{1}{2}}$ an; das wäre in unserer Bezeichnungsweise $2\left(\frac{u_M d}{\nu}\right)^2 \frac{d}{R_1} = 5832$; zu berücksichtigen ist allerdings, daß dieser Wert nur mit einer Sicherheit von etwa 2,5% gilt. An und für sich ist diese Genauigkeit ausreichend, und wir stellen fest, daß unser — genauerer — Wert 5740 innerhalb der Genauigkeitsgrenzen des Deanschen liegt. In diesem Sinn haben wir eine Übereinstimmung des kritischen Werts mit der früheren Angabe von W. R. DEAN. Dasselbe gilt für die kritische Wirbeldicke, die wir mit $\lambda/2 = 0{,}79\,d$ $(\sigma = 4)$ angeben. Der von YIH und SANGSTER [3] angegebene kritische Wert $S_{1\,\mathrm{Min}} = 600$ wird als falsch erkannt.

Zu den Methoden, die in den erwähnten Arbeiten verwandt werden, ist folgendes zu sagen: YIH und SANGSTER setzen die Eigenfunktion u als Fourier-Reihe an, lösen dann (3.2) und setzen das so gewonnene v in (3.1) ein; homogene Bestimmungsgleichungen für die noch unbekannten Koeffizienten des Ansatzes ergibt der Vergleich der Koeffizienten der nach trigonometrischen Funktionen entwickelten Gleichung (3.1). Die Determinante dieses Systems liefert, gleich Null gesetzt, eine Bedingungsgleichung für den gesuchten Eigenwert. Die Verfasser berücksichtigen von dieser Determinante unendlich hoher Ordnung nur ein Glied und berufen sich dabei auf CHANDRASEKHAR [*10*]. Tatsächlich führt jedoch CHANDRASEKHAR, der die Taylorschen rotierenden Zylinder so untersucht hat und zu brauchbaren Approximationen gelangte, den Vergleich in anderer Weise durch. Hinzu kommt, daß die Beschränkung auf ein Glied bei YIH und SANGSTER letztlich darauf hinausläuft, die Eigenfunktion $u(\eta)$ durch $C_1 \sin \pi\eta$ zu ersetzen. Wie Fig. 4 zeigt, besteht damit kein Grund zu der Hoffnung, so zu einem brauchbaren Resultat zu gelangen.

Es ist noch der Bemerkung wert, daß bereits DEAN [2] mit Fourier-Ansätzen für die gesuchten Eigenfunktionen arbeitet, die ebenfalls auf eine unendliche Determinante führen. DEAN treibt die Annäherung dieser Determinante durch solche endlicher Ordnung jedoch weit genug (bis zu 7reihigen), um seine numerischen Angaben sichern zu können.

Wir können unser Resultat noch mit dem kritischen Wert einer laminaren Grenzschichtströmung vergleichen. Fassen wir die halbe Kanalbreite als Grenzschichtdicke $\delta = d/2$ auf, so rechnet sich der kritische Wert der Kanalströmung um zu $2\left(\frac{u_M \delta}{\nu}\right)^2 \frac{\delta}{R}\Big|_{\mathrm{Min}} = 717$. Für laminare Grenzschichten ergab sich in entsprechender Approximation bei parabolischem Grundprofil der Wert 80, der beträchtlich niedriger liegt. In der Grenzschicht waren eben gerade diejenigen Wirbel die gefährlichsten, die eine große Dicke hatten und auch weit über die Grenzschicht hinausreichten. Das Vorhandensein der inneren Wand verbietet bei der Kanalströmung solche Wirbel; wir haben keine Grenzschicht im obigen Sinne, sondern diese wird von der konvexen Wand her stabilisierend beeinflußt. Der kritischen Wirbeldicke $\lambda/2 = 0{,}79\,d$ entsprach im Fall der Grenzschicht ein

kritischer Wert $2\,\mathrm{Re}^2\delta/R = 275$, der, verglichen mit 717, ebenfalls die stabilisierende Wirkung der inneren Kanalwand zum Ausdruck bringt. In unveröffentlichten Rechnungen fand H. WITTING für die Sekundärinstabilität einer ebenen Kanalströmung den kritischen Wert $2\,\mathrm{Re}^2\delta/R = 822$, wobei mit δ die halbe Kanalbreite bezeichnet ist, also einen Wert, der in der Nähe von 717 liegt.

Zusammenfassung

Die Wirbelinstabilität einer laminaren Strömung durch einen gekrümmten, unendlich hohen Kanal, dessen Breite klein ist gegenüber den Krümmungsradien der Kanalwände, wird durch die beiden Differentialgleichungen (2) beschrieben. Das System (2) stellt ein Eigenwertproblem dar, bei dem vor allem die kleinsten Eigenwerte S_1 in Abhängigkeit von dem Parameter σ mit den zugehörigen Eigenfunktionen interessieren; dabei beziehen wir uns bereits auf (3), wo neutrale Störungen ($\tau = \sigma$) betrachtet werden, und $S_1(\sigma)$ ist bis auf einen Zahlenfaktor die kritische Kurve des Parameters $2\,\mathrm{Re}^2\, d/R_1$ in Abhängigkeit von der Dicke $\lambda/2$ der angesetzten Wirbel. Mit Hilfe Greenscher Funktionen werden die Differentialgleichungen (3) in die Integralgleichungen (4) verwandelt, die es zunächst erlauben, untere Schranken für $S_1(\sigma)$ anzugeben (Fig. 2). Das Iterationsverfahren nach [9] liefert $S_1(\sigma)$ mit hinreichender Genauigkeit nach wenigen Iterationsschritten (Fig. 3). Ebenso ergeben sich damit die Eigenfunktionen (Fig. 4), also die Wirbelkomponenten, so daß in Fig. 5 das Aussehen der neutralen Wirbel angegeben werden kann. Der kritische Wert — aus dem kleinsten unter den Eigenwerten $S_1(\sigma)$ — ist $2\,\mathrm{Re}^2 d/R_1 = 5740$, während DEAN [2] 1928 den ebenfalls richtigen Wert $5832 \pm 2{,}5\,\%$ gefunden hatte. Die zugehörige kritische Wirbeldicke ist $\lambda/2 = 0{,}79\,d$. Die Ergebnisse von YIH und SANGSTER [3] erweisen sich als falsch. Die Arbeit verfolgte zwei Ziele: Eine Neubehandlung der Differentialgleichungen (5) mit exakt begründeten Methoden, um eine Beurteilung der Ergebnisse von DEAN und von YIH und SANGSTER möglich zu machen. Dann die Angabe der Eigenfunktionen, die es gestatten, das Aussehen der entstehenden Wirbel zu beschreiben.

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Determination of the Vorticity and the Gradients of Flow Parameters behind a Three-Dimensional Unsteady Curved Shock Wave

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1. Statement of the problem

The differential effects of shock fronts have been the subject of several researches of gradually increasing generality. My recent papers on three-dimensional flows of perfect gases calculated the jumps in the pressure, density and velocity gradients across shocks that are stationary [*1*, *2*] or pseudo-stationary [*3*]. The determination of vorticity in these two cases led to characterization of shock surfaces which may join two regions of irrotational flow and, in the latter case, to a theorem regarding the mode of propagation of such shocks.

By a different method, LIGHTHILL [*4*] derived an expression for the vorticity behind a steady shock in a gas obeying an arbitrary equation of state. HAYES [*5*] remarked that for plane shocks this result is contained in an earlier analysis of TRUESDELL [*6*], who concluded that *the magnitude of the vorticity generated by a shock of given strength and curvature depends only on the magnitude of the tangential component of the velocity and is independent of the form of the equation of state.* The strength of the shock is defined as the density jump divided by the density in front. As HAYES observed, TRUESDELL's statement suggests that the result is purely dynamical and thus should be derivable without recourse to any thermodynamic law, and HAYES has provided a proof based on the momentum equation alone. He did not mention, however, that this same independence may be inferred also from TRUESDELL's analysis. The reason the energy equation appears there is that, as in earlier work of THOMAS [*7*] and in my papers cited above, it is used to calculate the jumps of the full set of velocity gradients. By following the details in all these derivations of more complete results [*1*, *2*, *3*, *6*, *7*], it is easy to verify that in them, too, the energy equation is not used to calculate the vorticity jump.

The purposes of this paper are

1. To calculate the jumps of pressure, density, and velocity gradients in *full generality*, that is, for possibly unsteady shocks in fluids obeying *an arbitrary equation of state*. As usual, dissipative mechanisms such as viscosity and heat conduction are assumed absent.

2. To arrange the analysis, which follows and extends the methods used in several earlier papers [*1*, *2*, *3*, *6*, *7*], in such a way as to make plain *the independence of the jump in vorticity from the equation of energy and from the form of the equation of state*. In particular, the following results show that if a shock wave were possible in an inviscid but thermally conducting gas, the vorticity it generated would be independent of the thermal conductivity.

For the case when a curved shock is propagating into a quiet region, the results reduce to a simpler form. In parallel propagation, when the normal to the shock is steady, as is the case for example for spherical waves having a common center, the flow behind the shock remains irrotational.

The paper ends with a simpler derivation of the extended geometrical and kinematical conditions of compatibility recently discussed by THOMAS [*8*].

2. Notation and assumptions

As in previous works [*1*, *2*, *3*], we assume the shock surface $\Sigma(t)$ represented by a continuously differentiable function $x_i = x_i(y^1, y^2, t)$, where the x_i are rectangular Cartesian coordinates in an inertial frame and where the y^α are coordinates on $\Sigma(t)$. The range of Latin indices, referring to spatial tensors, is 1, 2, 3; that of Greek indices, referring to surface tensors, is 1, 2. The unit normal ξ_i, assumed continuously differentiable, is directed downstream.

A quantity f if evaluated on the upstream side of the shock will be denoted by f_1; if on the downstream side, by f; and $[f] \equiv f - f_1$.

3. Basic equations expressing balance of mass and momentum

The differential equation expressing the balance of momentum in an inviscid fluid subject to no extrinsic force is

(1) $$\varrho \frac{\partial u_i}{\partial t} + \varrho u_{i,j} u_j + p_{,i} = 0.$$

Let G denote any speed and ξ_i any direction, and let $\delta/\delta t$ denote the time derivative as apparent to an observer moving with velocity $G\xi_i$, *viz.*,

(2) $$\frac{\delta f}{\delta t} = \frac{\partial f}{\partial t} + G f_{,i} \xi_i.$$

Then the momentum equation (1) may be put into the form

(3) $$\varrho \frac{\delta u_i}{\delta t} + \varrho u_{i,j}(u_j - G\xi_j) + p_{,i} = 0.$$

In what follows, we use this result with $G\xi_i$ taken as the velocity of propagation of the shock.

The strength δ of the shock is defined by

(4) $$\delta = \frac{[\varrho]}{\varrho_1}.$$

Write $u_n = u_i \xi_i$; $u_{1n} = u_{1i}\xi_i$. Jump conditions expressing conservation of mass and momentum are

(5) $$[u_i] = -\frac{\delta}{1+\delta}(u_{1n} - G)\,\xi_i,$$

(6) $$[p] = \frac{\delta}{1+\delta}\varrho_1 (u_{1n} - G)^2.$$

Quantities expressed as functions of $\varrho_1, p_1, u_{1n}, \xi_i, G, \delta$, the metric tensor of the shock surface, and their derivatives along the shock surface we shall call *effectively calculated*. Such quantities depend only upon the shape, speed, and strength of the shock and upon flow conditions on its upstream side. The objective of differential shock analysis is effective calculation of the jumps in various gradients.

4. Properties of shocks following from the balance of mass and momentum alone

Differentiating both sides of (5) and (6) with respect to the coordinates on the shock surface, we have

$$u_{i,j}\, x_{j\alpha} = u_{1i,j}\, x_{j\alpha} - \left\{\frac{\delta}{1+\delta}(u_{1n} - G)\,\xi_i\right\}_{,\alpha} \equiv A^*_{i\alpha}, \tag{7}$$

$$p_{,j}\, x_{j\alpha} = p_{1,j}\, x_{j\alpha} + \left\{\frac{\delta}{1+\delta}\varrho_1 (u_{1n} - G)^2\right\}_{,\alpha} \equiv B^*_\alpha, \tag{8}$$

where $x_{i\alpha} \equiv \partial x_i / \partial y^\alpha$. The quantities $A^*_{i\alpha}$ and B^*_α are effectively calculated.

From (3) and (8) we eliminate $p_{,i}$ and obtain

$$u_{i,j}(u_j - G\,\xi_j)\, x_{i\alpha} = -\frac{B^*_\alpha}{\varrho} - \frac{\delta u_i}{\delta t}\, x_{i\alpha}. \tag{9}$$

We set

$$C_{i\alpha} \equiv g_{\alpha\alpha}\, x_{i\alpha}, \qquad C_{i3} \equiv \frac{u_i - G\,\xi_i}{u - G_n}, \qquad \alpha \text{ unsummed}, \tag{10}$$

where $g_{\alpha\beta} = (a_{\alpha\beta})^{-\frac{1}{2}}$, $a_{\alpha\beta}$ being the surface metric. From (5) it follows that C_{ij} is effectively calculated. The spatial unit tangent vectors to the surface coordinate curves are $x_{i\alpha}\, g_{\alpha\alpha}$, α unsummed. Thus $\det C_{ij} = 1$; therefore the matrix C_{ij} has an inverse D_{ij}:

$$D_{ij} = \text{cofactor of } C_{ji} \text{ in } \|C_{pq}\|; \qquad D_{ik}\, C_{kj} = \delta_{ij}, \qquad D_{ki}\, C_{jk} = \delta_{ij}. \tag{11}$$

Now set

$$B_{ij} \equiv u_{l,m}\, C_{li}\, C_{mj}. \tag{12}$$

By (11) we may invert this relation and obtain

$$u_{l,m} = B_{ij}\, D_{il}\, D_{jm}. \tag{13}$$

From (11), (10) and (5) we see that D_{ij} is effectively calculated. Thus if B_{ij} is effectively calculated, by (13) it follows that $u_{l,m}$ is effectively calculated.

Now from (12), (7) and (9) we find that

$$\begin{aligned} B_{\alpha\beta} &= A^*_{l\beta}\, x_{l\alpha}\, g_{\alpha\alpha}\, g_{\beta\beta}, && \alpha, \beta \text{ unsummed}, \\ B_{3\alpha} &= g_{\alpha\alpha}\, A^*_{i\alpha} \frac{u_i}{u_n}, && \alpha \text{ unsummed}, \\ B_{\alpha 3} &= -\frac{1}{u_n - G}\left(\frac{B^*_\alpha}{\varrho} + \frac{\delta u_i}{\delta t}\right). \end{aligned} \tag{14}$$

Thus the components $B_{\alpha\beta}$, $B_{\alpha 3}$ and $B_{3\alpha}$ are effectively calculated. For B_{33}, however, we obtain only

$$B_{33} = u_{l,m} \frac{(u_l - G\,\xi_l)\,(u_m - G\,\xi_m)}{(u_n - G)^2}, \tag{15}$$

and this does not provide effective calculation of B_{33}.

5. The vorticity jump

By (13) and (11) we calculate the vorticity behind the shock:

$$w_i = e_{ijk} u_{k,j} = e_{ijk} B_{lm} D_{lk} D_{mj} = -C_{ik} e_{kmp} B_{mp}. \tag{16}$$

Thus the vorticity is determined by $B_{mp} - B_{pm}$, and since these quantities are effectively calculated by (14), it follows that *the vorticity behind the shock is effectively calculated.* For steady or pseudo-steady shocks, both the method and the result reduce to those obtained in the earlier investigations [*1, 2, 3, 6*] described in § 1, and thus it is clear that *the energy equation and the equation of state are not required for determining the vorticity jump across a shock of a given strength.*

6. Use of the balance of energy to determine B_{33} and δ

In order to calculate effectively the full set of gradients $u_{i,j}$ behind the shock, we need to determine B_{33}, and to this end we introduce an equation of state $f(\varepsilon, \varrho, \eta) = 0$, where ε is the specific internal energy and η is the specific entropy. This equation is assumed soluble for any of its variables as a function of the other two, and the resulting equation is assumed continuously differentiable. The pressure p occuring in (1) and (6) is assumed to be related to the equation of state as follows: $p = -\partial\varepsilon/\partial(1/\varrho)$, where $\eta = \text{const}$. The speed of sound, c, is defined formally as $c^2 = (\partial p/\partial\varrho)_\eta$. We assume further that the fluid does not conduct heat. When (1) and the equation of continuity are satisfied, the equation of energy is equivalent to

$$\frac{\partial p}{\partial t} - \varrho\, u_i \frac{\partial u_i}{\partial t} - \varrho\, u_i u_j u_{i,j} + c^2 \varrho\, u_{k,k} = 0. \tag{17}$$

By use of (2) we may put this condition into the equivalent form

$$\varrho\, u_{j,k} (u_j - G\,\xi_j)\,(u_k - G\,\xi_k) = \frac{\delta p}{\delta t} - \varrho \frac{\delta u_k}{\delta t} (u_k - G\,\xi_k) + c^2 \varrho\, u_{k,k}. \tag{18}$$

The quantity $u_{k,k}$ may be eliminated by (13):

$$\begin{aligned} u_{k,k} &= B_{ij} D_{ik} D_{jk} \\ &= B_{11}(1+\chi_1^2) + B_{22}(1+\chi_2^2) + 2B_{(12)}\chi_1\chi_2 - 2B_{(13)}\chi_1 - 2B_{(23)}\chi_2 + B_{33}, \end{aligned} \tag{19}$$

where $B_{ij} \equiv \frac{1}{2}(B_{ij} + B_{ji})$, and

$$\chi_\alpha \equiv \frac{v_\alpha g_{\alpha\alpha}}{u_n - G}, \qquad \alpha \text{ unsummed}, \quad v_\alpha = u_i x_{i\alpha}. \tag{20}$$

From (15), (18) and (19) we observe that B_{33} is effectively calculated. We have thus determined all the components of the tensor B_{ij}.

Moreover, conservation of energy at the shock implies

$$[h] = \frac{\delta\{1+\frac{1}{2}\delta\}}{(1+\delta)^2}(u_{1n}-G)^2, \tag{21}$$

where h is the enthalpy, $h \equiv \varepsilon + p/\varrho$. Since $h = h(p, \varrho)$, a formula for δ as a function of $(u_{1n}-G)$, p_1, ϱ_1, may be obtained in principle by elimination among equations (21), (4) and (6). For the special case of a perfect gas with constant specific heats, there thus results

$$\delta = \frac{2\{\varrho_1(u_{1n}-G)^2 - \gamma p_1\}}{2\gamma p_1 + (\gamma-1)\varrho_1(u_{1n}-G)^2}, \tag{22}$$

where $\gamma = c_p/c_v$.

7. Evaluation of the pressure and density gradients

So as to calculate the pressure gradient behind the shock wave, we substitute for $u_{i,j}$ in equation (3) and obtain

$$p_{,i} = -\varrho(u_n - G)\, B_{j3} D_{ji} - \varrho \frac{\delta u_i}{\delta t}. \tag{23}$$

To evaluate the density gradient we differentiate both sides of the equation (4) with respect to the coordinates y^α and get

$$\varrho_{,i} x_{i\alpha} = \varrho_{1,i} x_{i\alpha} + (\delta\varrho_1)_{,\alpha} = C^*_\alpha. \tag{24}$$

The equation of continuity gives

$$\frac{\delta\varrho}{\delta t} + \varrho_{,k}(u_k - G\xi_k) + \varrho u_{k,k} = 0. \tag{25}$$

The equations (24) and (25) can be written as

$$\varrho_{,j} C_{ji} = d_i, \tag{26}$$

where

$$d_1 = g_{11} C^*_1, \quad d_2 = g_{22} C^*_2, \quad d_3 = -\left(\varrho u_{k,k} + \frac{\delta\varrho}{\delta t}\right)\frac{1}{u_n - G}.$$

Hence by inversion we have

$$\varrho_{,j} = d_i D_{ij}. \tag{27}$$

8. Calculation of the quantities B_{ij} and d_i

As in the case of steady and pseudo-steady flows, it is possible to calculate explicitly the quantities B_{ij} and d_i in the case of unsteady flow of a perfect gas with constant specific heats, if we assume that the flow upstream of the shock is uniform. For simplicity we take the lines of curvature as the coordinate curves on the shock surface. Thus

$$a_{12} = b_{12} = 0; \quad K_1 = b_{11}/a_{11}, \quad K_2 = b_{22}/a_{22}, \tag{28}$$

where $b_{\alpha\beta}$ are the components of the second fundamental form and K_1 and K_2 are the principal normal curvatures of the shock surface. WEINGARTEN'S formulae

assert that

(29) $$\xi_{i,\alpha} = - a^{\gamma\beta} b_{\alpha\gamma} x_{i\beta} = - K_{\alpha} x_{i\alpha}, \qquad \alpha \text{ unsummed},$$

where $a^{\alpha\beta}$ are the components of the tensor reciprocal to $a_{\alpha\beta}$. We further have the relation [5, 8]

(30) $$\frac{\delta \xi_i}{\delta t} = - a^{\alpha\beta} G_{,\alpha} x_{i\beta}.$$

Keeping these results and the assumption of uniform flow upstream of the shock in mind, we get

(31) $$A^*_{i\alpha} = - \left\{ \frac{B^*_\alpha \xi_i}{\varrho_1 (u_{1n} - G)} + \frac{[\varrho]}{\varrho_1 (u_{1n} - G)^2} \left(- (u_{1n} - G) K_\alpha x_{i\alpha} + \xi_i (v_\alpha K_\alpha + G_{,\alpha}) \right) \right\},$$

(32) $$B^*_\alpha = - \frac{4 \varrho_1 (u_{1n} - G)(v_\alpha K_\alpha + G_{,\alpha})}{\gamma + 1},$$

(33) $$C^*_\alpha = - \frac{4 \varrho_1^2 \gamma (\gamma + 1)(u_{1n} - G)(v_\alpha K_\alpha + G_{,\alpha}) p_1}{\{2 \gamma p_1 + (\gamma - 1) \varrho_1 (u_{1n} - G)^2\}^2},$$

(34) $$\frac{\delta u_i}{\delta t} = - \frac{\delta p}{\delta t} \frac{\xi_i}{\varrho_1 (u_{1n} - G)} - \frac{\delta}{1 + \delta} \left\{ - (u_{1n} - G) a^{\alpha\beta} G_{,\alpha} x_{i\beta} + \left(v^\alpha G_{,\alpha} + \frac{\delta G}{\delta t} \right) \xi_i \right\},$$

(35) $$\frac{\delta p}{\delta t} = - \frac{4 \varrho_1 (u_{1n} - G) \left(v^\alpha G_{,\alpha} + \frac{\delta G}{\delta t} \right)}{\gamma + 1},$$

(36) $$\frac{\delta \varrho}{\delta t} = - \frac{4 \varrho_1^2 \gamma (\gamma + 1)(u_{1n} - G) \left(v^\alpha G_{,\alpha} + \frac{\delta G}{\delta t} \right) p_1}{\{2 \gamma p_1 + (\gamma - 1) \varrho_1 (u_{1n} - G)^2\}^2}.$$

The quantities B_{ij} and d_i can now be readily calculated and are given as

$$B_{11} = \delta (u_n - G) K_1,$$

$$B_{12} = B_{21} = 0,$$

$$B_{22} = \delta (u_n - G) K_2,$$

$$B_{13} = a_{11} \left\{ \frac{4 (v_1 K_1 + G_{,1})}{\gamma + 1} - \delta G_{,1} \right\},$$

(37) $$B_{23} = a_{22} \left\{ \frac{4 (v_2 K_2 + G_{,2})}{\gamma + 1} - \delta G_{,2} \right\},$$

$$B_{31} = a_{11} \left\{ \frac{4 (v_1 K_1 + G_{,1})}{\gamma + 1} + \delta v_1 K_1 - \frac{\delta}{1 + \delta} (v_1 K_1 + G_{,1}) \right\},$$

$$B_{32} = a_{22} \left\{ \frac{4 (v_2 K_2 + G_{,2})}{\gamma + 1} + \delta v_2 K_2 - \frac{\delta}{1 + \delta} (v_2 K_2 + G_{,2}) \right\},$$

$$B_{33} = \frac{\gamma p}{\varrho (u_n - G)} u_{k,k} - \left\{ \delta G_{,\alpha} v^\alpha - \left(\frac{\delta}{1 + \delta} + \frac{\gamma}{\gamma + 1} \right) \left(v^\alpha G_{,\alpha} + \frac{\delta G}{\delta t} \right) \right\} \frac{1}{u_n - G},$$

and

$$d_1 = -\frac{4a_{11}\varrho_1^2\gamma(\gamma+1)(u_{1n}-G)(v_1K_1+G_{,1})p_1}{\{2\gamma p_1+(\gamma-1)\varrho_1(u_{1n}-G)^2\}^2},$$

(38) $$d_2 = -\frac{4a_{22}\varrho_1^2\gamma(\gamma+1)(u_{2n}-G)(v_2K_2+G_{,2})p_1}{\{2\gamma p_1+(\gamma-1)\varrho_1(u_{1n}-G)^2\}^2},$$

$$d_3 = -\frac{1}{u_\alpha-G}\left\{\varrho\, u_{k,k}+\frac{4\varrho_1^2\gamma(\gamma+1)(u_{1n}-G)\left(v^\alpha G_{,\alpha}+\frac{\delta G}{\delta t}\right)p_1}{\{2\gamma p_1+(\gamma-1)\varrho_1(u_{1n}-G)^2\}^2}\right\},$$

where the quantity $u_{k,k}$ entering the expressions for B_{33} and d_3 is given by the equation (19).

Thus when the flow upstream of the shock is uniform the expression for the vorticity vector behind the shock is

(39) $$w_i = a_{11}a_{22}\frac{\delta^2}{1+\delta}\{-(v_1K_1+G_{,1})x_{i2}+(v_2K_2+G_{,2})x_{i1}\},$$

which agrees with HAYES' result.

Now if the shock propagates into still air, the values of B_{ij} and d_i take a very simple form. In that case we notice from the expressions (37) that *in the parallel propagation of shock waves*, i.e., *the normal directions to the shock surface* $\Sigma(t)$ *at* $t=t_0$ *remain normal to* $\Sigma(t)$ *for* $t>t_0$, e.g. *in the case of plane parallel shocks or spherical shocks having common center, the flow behind them remains irrotational.*

Having determined the gradients of flow parameters *we can determine their variation along the shock wave* as was done in the stationary case [2].

9. Derivation of Thomas' "geometrical and kinematical conditions of compatibility"

Finally we give a simpler derivation of the "first extended compatibility conditions" defined by THOMAS [8]. In fact, if we are given a jump relation

(40) $$[Z] = A$$

and we differentiate both sides of this relation with respect to y^α, we get

(41) $$[Z_{,i}]\, x_{i\alpha} = A_{,\alpha}.$$

Multiplying both sides by $a^{\alpha\beta}x_{j\beta}$ and using the relation $a^{\alpha\beta}x_{i\alpha}x_{i\beta}=\delta_{ij}-\xi_i\xi_j$ yields

(42) $$[Z_{,i}] = \{[Z_{,j}]\,\xi_j\}\,\xi_i + a^{\alpha\beta}A_{,\alpha}x_{i\beta}.$$

This is THOMAS' "first geometrical condition of compatibility".

Applying the operator $\delta/\delta t$ to both sides of (40), we get

(43) $$\left[\frac{\delta Z}{\delta t}\right] = -\{[Z_{,i}]\,\xi_i\}\,G + \frac{\delta A}{\delta t},$$

which is THOMAS' "first kinematical condition of compatibility".

THOMAS' "second conditions of compatibility" can be derived from the first ones, as he has remarked.

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On the Convergence of the Rayleigh Quotient Iteration for the Computation of the Characteristic Roots and Vectors. I

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1. Let $A = (a_{\mu\nu})$ be an $(n \times n)$ real symmetric matrix. Then for a vector $\xi = (x_1, \ldots, x_n)$ we put

(1) $$Q_A(\xi) = \xi A \xi' = \sum_{\mu,\nu} a_{\mu\nu} x_\mu x_\nu .$$

The quotient

(2) $$\frac{Q_A(\xi)}{|\xi|^2}$$

is called the *Rayleigh quotient* corresponding to ξ. If ξ is a characteristic vector belonging to a characteristic root λ, then the corresponding Rayleigh quotient is λ. Therefore the following procedure has been devised for obtaining a sequence of numbers $\lambda_\varkappa$ $(\varkappa = 0, 1, \ldots)$ converging to a characteristic root:

For any $\lambda_\varkappa$ of the sequence $(\varkappa = 0, 1, \ldots)$ find an approximate solution $\xi_\varkappa$ of the homogeneous system

(3) $$A \xi_\varkappa' = \lambda_\varkappa \xi_\varkappa'$$

and put

(4) $$\lambda_{\varkappa+1} = \frac{Q_A(\xi_\varkappa)}{|\xi_\varkappa|^2} \qquad (\varkappa = 0, 1, \ldots).$$

My attention was drawn to this method by JOHN TODD, who used it in his lectures as long ago as 1945. It appears to converge fairly well in numerical practice. In what follows I give some theoretical results on the convergence of this method.

2. The crucial point in the discussion of the above method is of course a suitable rule for the computation of the "approximate solution" $\xi_\varkappa$ of (3). The rule I shall use in the first part of this discussion consists in taking an arbitrary vector $\eta \neq 0$ and in putting

(5) $$\xi_\varkappa' = (A - \lambda_\varkappa E)^{-1} \eta' .$$

The theoretical arguments in support of this rule are given in another paper**.

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* In writing this paper I had very valuable discussions with Mr. CHR. BLATTER.

** A. OSTROWSKI, „Über näherungsweise Auflösung von Systemen homogener linearer Gleichungen", Journal of Applied Mathematics and Physics (ZAMP), Basle, Vol. **8** (1957), pp. 280—285.

Since the formulae (4), (5) are *invariant*, for our discussion we can introduce *normal coordinates* from the beginning and therefore without loss of generality put

$$Q_A(\xi) = \sum_{\nu=1}^{n} \mu_\nu x_\nu^2, \tag{6}$$

where

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \tag{7}$$

are the characteristic roots of A, ordered increasingly. Then, if

$$\eta = (y_1, \ldots, y_n), \tag{8}$$

we have from (5)

$$\xi_\varkappa = (x_1^{(\varkappa)}, \ldots, x_n^{(\varkappa)}) = \left(\frac{y_1}{\mu_1 - \lambda_\varkappa}, \ldots, \frac{y_n}{\mu_n - \lambda_\varkappa}\right),$$

$$Q_A(\xi_\varkappa) = \sum_{\nu=1}^{n} \frac{\mu_\nu y_\nu^2}{(\mu_\nu - \lambda_\varkappa)^2}, \qquad |\xi_\varkappa|^2 = \sum_{\nu=1}^{n} \frac{y_\nu^2}{(\mu_\nu - \lambda_\varkappa)^2},$$

and finally

$$\lambda_{\varkappa+1} = \frac{\sum\limits_{\nu=1}^{n} \frac{\mu_\nu y_\nu^2}{(\mu_\nu - \lambda_\varkappa)^2}}{\sum\limits_{\nu=1}^{n} \frac{y_\nu^2}{(\mu_\nu - \lambda_\varkappa)^2}} \qquad (\varkappa = 0, 1, \ldots). \tag{9}$$

3. The expression on the right side in (9), if all products $\mu_\nu y_\nu^2$ in the numerator are replaced by $\mu_1 y_\nu^2$ or $\mu_n y_\nu^2$, reduces to μ_1 and μ_n, respectively. We see that in any case

$$\mu_1 \leq \lambda_\varkappa \leq \mu_n \qquad (\varkappa = 1, \ldots). \tag{10}$$

In the expression on the right side of (9) a μ_ν drops out if the corresponding y_ν vanishes. Denote the remaining *distinct* μ_ν in increasing order by

$$\sigma_1 < \sigma_2 < \cdots < \sigma_m. \tag{11}$$

Then the formula (9) becomes

$$\lambda_{\varkappa+1} = \frac{\sum\limits_{\mu=1}^{m} \frac{\sigma_\mu p_\mu}{(\sigma_\mu - \lambda_\varkappa)^2}}{\sum\limits_{\mu=1}^{m} \frac{p_\mu}{(\sigma_\mu - \lambda_\varkappa)^2}} \qquad (\varkappa = 0, 1, \ldots), \tag{12}$$

where the coefficients p_μ are all *positive.*

4. Denote one of the σ_μ by σ and the corresponding p_μ by p. By subtracting σ from both sides of (12) we obtain

$$\lambda_{\varkappa+1} - \sigma = \frac{\sum\limits_{\mu=1}^{m} p_\mu \frac{\sigma_\mu - \sigma}{(\sigma_\mu - \lambda_\varkappa)^2}}{\sum\limits_{\mu=1}^{m} \frac{p_\mu}{(\sigma_\mu - \lambda_\varkappa)^2}} = (\sigma - \lambda_\varkappa)^2 \frac{\sum\limits_{\mu=1}^{m}{}' p_\mu \frac{\sigma_\mu - \sigma}{(\sigma_\mu - \lambda_\varkappa)^2}}{p + \sum\limits_{\mu=1}^{m}{}' p_\mu \left(\frac{\sigma - \lambda_\varkappa}{\sigma_\mu - \lambda_\varkappa}\right)^2}, \tag{13}$$

where in the last sums the terms with the index μ for which $\sigma_\mu = \sigma$ are to be omitted. From this formula we see that if one $\lambda_\varkappa$ gets sufficiently near to σ,

the whole sequence $\lambda_\varkappa$ tends to σ. Then dividing the first and the last term of (13) by $(\lambda_\varkappa-\sigma)^2$, we obtain

(14) $$\frac{\lambda_{\varkappa+1}-\sigma}{(\lambda_\varkappa-\sigma)^2}\to\frac{1}{p}\sum_{\mu=1}^{m}{}'\frac{p_\mu}{\sigma_\mu-\sigma}\qquad(\lambda_\varkappa\to\sigma).$$

We see that in this case the convergence is at least *quadratic*. The convergence could even be faster than that, if the limit in (14) were 0. However, for $\sigma=\sigma_1$ or $\sigma=\sigma_m$ the convergence is exactly quadratic*.

5. In order to characterize the convergence neighbourhood of σ put

(15) $$d=\operatorname*{Min}_{\sigma_\mu\neq\sigma}|\sigma_\mu-\sigma|,\qquad P=\sum_{\mu=1}^{m}p_\mu.$$

If we then assume

(16) $$|\lambda_\varkappa-\sigma|\leq\frac{d}{2},$$

for $\sigma_\mu\neq\sigma$ it follows that

$$\frac{|\sigma_\mu-\sigma|}{(\sigma_\mu-\lambda_\varkappa)^2}=\frac{|\sigma_\mu-\sigma|}{(\sigma_\mu-\sigma+\sigma-\lambda_\varkappa)^2}\leq\frac{|\sigma_\mu-\sigma|}{\left|\frac{\sigma_\mu-\sigma}{2}\right|^2}=\frac{4}{|\sigma_\mu-\sigma|}\leq\frac{4}{d}.$$

Introducing this in the numerator of the last term in (13) and replacing the denominator by p, we get

$$\frac{|\lambda_{\varkappa+1}-\sigma|}{(\lambda_\varkappa-\sigma)^2}\leq\frac{4}{d}\,\frac{\sum\limits_{p_\mu\neq p}p_\mu}{p}=\frac{4}{d}\,\frac{P-p}{p},$$

(17) $$\frac{|\lambda_{\varkappa+1}-\sigma|}{(\lambda_\varkappa-\sigma)^2}\leq\frac{4}{d}\left(\frac{P}{p}-1\right),$$

(18) $$\left|\frac{\lambda_{\varkappa+1}-\sigma}{\lambda_\varkappa-\sigma}\right|\leq\frac{4}{d}\,\frac{P-p}{p}\,|\lambda_\varkappa-\sigma|.$$

Therefore it follows that if for one index $\varkappa$ we have

(19) $$|\lambda_\varkappa-\sigma|<\frac{d}{2}\operatorname{Min}\left(\frac{p}{2(P-p)},1\right),$$

then the distances $|\lambda_\varkappa-\sigma|$ from that index on are strictly diminishing and therefore converge to 0. We see that a convergence neighborhood of σ is given by (19).

Of course in this way we obtain only sequences $\lambda_\varkappa$ converging to those characteristic values μ_ν of A which remain among the σ_μ. If the constant vector η is orthogonal to all characteristic vectors corresponding to a characteristic root μ_ν,

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* This result agrees with the note of Walter Kohn, "A Variational Iteration Method for Solving Secular Equations", Journal of Chemical Physics, **17**, 670 (1949). In this note W. Kohn discusses the application of the Rayleigh quotient method in taking one of the coordinate unit vectors for η. He says then (in our notation): "A more careful analysis shows that

$$\delta\lambda_{\varkappa+1}=K_\varkappa(\delta\lambda_\varkappa)^2,$$

where $K_\varkappa$ is in general of the order 1."

then this characteristic root drops out, since all corresponding y_ν^2 vanish. However, we obtain for each choice of η at least one characteristic root, if the starting λ_0 is chosen near enough to such a root. In particular cases it may be necessary to try out several choices of η, for instance to try each of the coordinate unit vectors.

6. In order to discuss the global convergence situation we have to consider all fixed points of the iteration $\lambda_{\varkappa+1} = \varphi(\lambda_\varkappa)$, where

$$\varphi(\lambda) = \frac{\sum_{\mu=1}^{m} \frac{\sigma_\mu p_\mu}{(\sigma_\mu - \lambda)^2}}{\sum_{\mu=1}^{m} \frac{p_\mu}{(\sigma_\mu - \lambda)^2}}. \tag{20}$$

The corresponding algebraic equation for the fixed points, $\lambda = \varphi(\lambda)$, becomes an equation of degree $\leqq 2m-1$. We know already m different roots of this equation, given by the σ_μ, and we have seen that all these fixed points are points of attraction. By a theorem which we proved in another communication*, between two consecutive fixed points of attraction there is always at least one fixed point of repulsion. We see that besides the m fixed points (11) the iteration by $\varphi(\lambda)$ has at least $m-1$ further different fixed points. Therefore the iteration by $\varphi(\lambda)$ has *exactly* the $2m-1$ fixed points

$$\sigma_1 < \vartheta_1 < \sigma_2 < \vartheta_2 < \cdots < \vartheta_{m-1} < \sigma_m, \tag{21}$$

where the ϑ_μ are points of repulsion.

In order to obtain an algebraic equation of degree $m-1$ satisfied by the ϑ_μ, subtract λ from the expression (20) and multiply by the denominator. Then we obtain

$$\sum_{\mu=1}^{m} \left[\frac{\sigma_\mu p_\mu}{(\lambda-\sigma_\mu)^2} - \frac{\lambda p_\mu}{(\lambda-\sigma_\mu)^2} \right] = -\sum_{\mu=1}^{m} \frac{p_\mu}{\lambda-\sigma_\mu}. \tag{22}$$

Therefore the polynomial equation for all fixed points is given by

$$\prod_{\mu=1}^{m} (\lambda-\sigma_\mu)^2 \sum_{\mu=1}^{m} \frac{p_\mu}{\lambda-\sigma_\mu} = 0, \tag{23}$$

while the polynomial equation satisfied by the ϑ_μ is obtained in the form

$$\prod_{\mu=1}^{m} (\lambda-\sigma_\mu) \sum_{\mu=1}^{m} \frac{p_\mu}{\lambda-\sigma_\mu} = 0. \tag{24}$$

The equation for the repulsive fixed points ϑ_μ, given above in normal coordinates, can be written in the invariant form

$$Q_{(\lambda E - A)^{-1}}(\eta) = 0. \tag{25}$$

7. Since the ϑ_μ are points of repulsion, we have $|\varphi'(\vartheta_\mu)| \geqq 1$. It can easily be shown that at each of the points ϑ_μ we have

$$\varphi'(\vartheta_\mu) \geqq 1. \tag{26}$$

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* A. OSTROWSKI, Mathematische Miszellen XXV, „Über das Verhalten von Iterationsfolgen im Divergenzfall", Jahresber. d. DMV, Bd. **59** (1956), pp. 69–79.

Indeed, if we take $\lambda > \sigma_\mu$, in a sufficiently small neighborhood of σ_μ we have

$$\varphi(\lambda) < \lambda.$$

If λ increases, this inequality remains valid until we get the first point for which $\varphi(\lambda) = \lambda$. This is ϑ_μ. We have therefore for sufficiently small positive ε: $\varphi(\vartheta_\mu - \varepsilon) < \vartheta_\mu - \varepsilon$. On the right side, replace ϑ_μ by $\varphi(\vartheta_\mu)$, subtract $\varphi(\vartheta_\mu)$ on both sides and divide by $-\varepsilon$. We obtain

$$\frac{\varphi(\vartheta_\mu - \varepsilon) - \varphi(\vartheta_\mu)}{-\varepsilon} > 1,$$

and from this, since $\varepsilon \downarrow 0$, (26) follows immediately.

8. The further discussion of the global convergence problem appears to present considerable difficulties if treated by the method of conjugate couples of points (*cf.* the paper cited in § 6). Indeed the determination of such couples depends on the solution of the equation

$$\lambda = \varphi(\varphi(\lambda)),$$

which reduces to an algebraic equation of degree $4m - 1$; we may expect as many as $m - 1$ couples of conjugate points.

Only when $m = 2$ is the problem solved immediately. Indeed it follows then from the analysis given in Section 7 that as soon as a $\lambda_\varkappa$ lies in one of the open intervals (σ_1, ϑ_1), (ϑ_1, σ_2), the sequence $\lambda_\varkappa$ converges to σ_1 in the first case and to σ_2 in the second. On the other hand it follows from (10) that λ_1 lies in $< \sigma_1, \sigma_2 >$. Therefore the decision in this case is possible after the first iterative step, as soon as we have determined ϑ_1. But here the equation (25) gives immediately for ϑ_1 the expression

$$\vartheta_1 = \frac{a_{11} y_2^2 - 2 a_{12} y_1 y_2 + a_{22} y_1^2}{y_1^2 + y_2^2}. \tag{27}$$

9. In the foregoing discussion, in order to obtain $\xi_\varkappa$ we used an arbitrary *fixed* vector η in (5). On the other hand, in the theory of direct iteration a variant due to H. WIELANDT * and called *broken iteration* is often used and consists in forming recursively the vectors $\xi_\varkappa$ given by

$$\xi_\varkappa = A^{-1} \xi_{\varkappa-1},$$

starting with an arbitrary vector $\xi_{-1} = \eta$. In combining this idea of broken iteration with our rule (4) we obtain the following modification of our rule:

For any $\lambda_\varkappa$ define a vector $\xi_\varkappa$ by

$$\xi_\varkappa = (A - \lambda_\varkappa E)^{-1} \xi_{\varkappa-1} \qquad (\varkappa = 0, 1, \ldots), \tag{28}$$

where ξ_{-1} is an arbitrary vector $\eta \neq 0$, and then put

$$\lambda_{\varkappa+1} = \frac{Q_A(\xi_\varkappa)}{|\xi_\varkappa|^2} \qquad (\varkappa = 0, 1, \ldots). \tag{29}$$

It will turn out in this case that the convergence is indeed considerably hastened, becoming *cubic* instead of quadratic.

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* H. WIELANDT, ,,Beiträge zur mathematischen Behandlung komplexer Eigenwertprobleme, V: Bestimmung höherer Eigenwerte durch gebrochene Iteration", Bericht B 44/J/37 der aerodynamischen Versuchsanstalt Göttingen, 1944.

10. In order to discuss this procedure we assume again without loss of generality that the coordinates are normal and that (6), (7), (8) hold. If we again put $\xi_\varkappa = (x_\nu^{(\varkappa)})$, it follows from (28) that $x_\nu^{(\varkappa)} = \dfrac{x_\nu^{(\varkappa-1)}}{\mu_\nu - \lambda_\varkappa}$, and therefore

$$x_\nu^{(k)} = \frac{y_\nu}{\prod\limits_{\varkappa=0}^{k} (\mu_\nu - \lambda_\varkappa)} \quad (k = 0, 1, \ldots). \tag{30}$$

If we put

$$N_{\nu,k} = \prod_{\varkappa=0}^{k} (\mu_\nu - \lambda_\varkappa) \qquad (\nu = 1, \ldots, n;\ k = 0, 1, \ldots),$$

we obtain

$$x_\nu^{(k)} = \frac{y_\nu}{N_{\nu,k}}, \qquad Q_A(\xi_k) = \sum_{\nu=1}^{n} \frac{\mu_\nu y_\nu^2}{N_{\nu,k}^2},$$

$$\lambda_{k+1} = \frac{\sum\limits_{\nu=1}^{n} \dfrac{\mu_\nu y_\nu^2}{N_{\nu,k}^2}}{\sum\limits_{\nu=1}^{n} \dfrac{y_\nu^2}{N_{\nu,k}^2}}. \tag{31}$$

Here we again disregard the μ_ν corresponding to the vanishing y_ν and denote the remaining distinct μ_ν by (11). Then (31) becomes, with appropriate positive p_μ:

$$\lambda_{k+1} = \frac{\sum\limits_{\mu=1}^{m} \dfrac{\sigma_\mu p_\mu}{M_{\mu,k}^2}}{\sum\limits_{\mu=1}^{m} \dfrac{p_\mu}{M_{\mu,k}^2}} \qquad (k = 0, 1, \ldots), \tag{32}$$

where

$$M_{\mu,k} = \prod_{\varkappa=0}^{k} (\sigma_\mu - \lambda_\varkappa) \qquad (\mu = 1, \ldots, m;\ k = 0, 1, \ldots). \tag{33}$$

11. Again denote by σ one of the σ_μ and by p, M_k the corresponding p_μ, $M_{\mu,k}$. Then from (32) follows

$$\lambda_{k+1} - \sigma = \frac{\sum\limits_{\mu=1}^{m}{}' \dfrac{(\sigma_\mu - \sigma) p_\mu}{M_{\mu,k}^2}}{\dfrac{p}{M_k^2} + \sum\limits_{\mu=1}^{m}{}' \dfrac{p_\mu}{M_{\mu,k}^2}}, \tag{34}$$

where in the sums $\sum'$ the terms with the index μ for which $\sigma_\mu = \sigma$ are to be omitted. From (34) putting

$$D_k = \frac{\sum\limits_{\mu=1}^{m}{}' \dfrac{(\sigma_\mu - \sigma) p_\mu}{M_{\mu,k}^2}}{p + \sum\limits_{\mu=1}^{m}{}' p_\mu \dfrac{M_k^2}{M_{\mu,k}^2}}, \tag{35}$$

we have again

$$\lambda_{k+1} - \sigma = D_k M_k^2. \tag{36}$$

Now put

$$d = \min_{\sigma_\mu \neq \sigma} |\sigma_\mu - \sigma|, \qquad P = \sum_{\mu=1}^{m} p_\mu, \qquad K = \frac{P}{p} (\sigma_m - \sigma_1), \tag{37}$$

and take a $\delta > 0$ such that

(38) $$\delta < \delta' = \mathrm{Min}\left(\frac{d}{2}, \frac{d^2}{4K}\right).$$

Then we have obviously

(39) $$\varepsilon \equiv \frac{\delta}{d-\delta} < 1,$$

$$K\varepsilon^2 = \delta \frac{K\delta}{(d-\delta)^2} < \delta \frac{d^2/4}{\left(d - \frac{d}{2}\right)^2},$$

(40) $$K\varepsilon^2 < \delta.$$

12. Suppose now that we have for $\varkappa = 0, 1, \ldots, k$

(41) $$|\lambda_\varkappa - \sigma| \leqq \delta \qquad (\varkappa = 0, 1, \ldots, k).$$

Then we have from (33), since $\sigma_\mu \neq \sigma$,

(42) $$|M_{\mu,k}| \geqq (d-\delta)^{k+1}$$

and therefore by (37) and (35), since all terms in the denominator of (35) are positive,

$$|D_k| \leqq \frac{1}{p} P \frac{\mathrm{Max}\,|\sigma_\mu - \sigma|}{(d-\delta)^{2k+2}},$$

$$|D_k| \leqq K(d-\delta)^{-2k-2}.$$

It follows then from (39) and (40)

(43) $$|D_k \delta^{2k+2}| \leqq K\varepsilon^{2k+2} < \varepsilon^{2k}\delta \qquad (k = 0, 1, \ldots).$$

We have now from (36), since $M_k^2 \leqq \delta^{2k+2}$ by (41),

(44) $$|\lambda_{k+1} - \sigma| \leqq \varepsilon^{2k}\delta \qquad (k = 0, 1, \ldots).$$

We see that the sequence λ_k is convergent to σ and we have for each k: $|\lambda_k - \sigma| \leqq \delta$, provided only that

(45) $$|\lambda_0 - \sigma| \leqq \delta.$$

13. We now prove that *if* $\lambda_k \to \sigma$ *and if none of the* λ_k *is equal to* σ, *then*

(46) $$\frac{\lambda_{k+1} - \sigma}{(\lambda_k - \sigma)^3} \to \gamma \qquad (\varkappa \to \infty),$$

where γ *is a positive constant equal to one of the quotients* $\dfrac{1}{(\sigma_\mu - \sigma)^2}$.

We assume first that (45) is satified. Observe that from (44) and (45) by definition of M_k we have

(47) $$M_k^2 \leqq \delta^2 \prod_{\varkappa=1}^{k} (\delta\varepsilon^{2(\varkappa-1)})^2 = \delta^{2k+2}\varepsilon^{2k(k-1)}.$$

On the other hand, if we divide both sides of (36) by $\lambda_k - \sigma$ and use (47), it follows that

$$\left|\frac{\lambda_{k+1} - \sigma}{\lambda_k - \sigma}\right| = |D_k M_k M_{k-1}| \leqq |D_k|\,\delta^2\left(\prod_{\varkappa=1}^{k-1}(\delta\varepsilon^{2(\varkappa-1)})^2\right)\delta\varepsilon^{2(k-1)}$$
$$= |D_k|\,\delta^{2k+2}\,\frac{1}{\delta}\,\varepsilon^{2(k-1)^2},$$

and therefore by (43)

$$\left|\frac{\lambda_{k+1}-\sigma}{\lambda_k-\sigma}\right| < \varepsilon^{k^2}. \tag{48}$$

On the other hand, if we write (36) for k and $k-1$ and divide, we obtain

$$\frac{\lambda_{k+1}-\sigma}{(\lambda_k-\sigma)^3} = \frac{D_k}{D_{k-1}}; \tag{49}$$

therefore we have to discuss D_k as given by (35).

14. In the formula (33) for $M_{\mu,k}$ the general factor $\sigma_\mu - \lambda_\varkappa$ can be written as $(\sigma_\mu-\sigma)\left(1+\frac{\sigma-\lambda_\varkappa}{\sigma_\mu-\sigma}\right)$. Therefore, putting

$$T_{\mu,k} = \prod_{\varkappa=0}^{k}\left(1+\frac{\sigma-\lambda_\varkappa}{\sigma_\mu-\sigma}\right), \tag{50}$$

we have

$$M_{\mu,k} = (\sigma_\mu-\sigma)^{k+1}\, T_{\mu,k}. \tag{51}$$

Since $\sum_{\varkappa=0}^{\infty}|\sigma-\lambda_\varkappa|$ is convergent by (48), we see that

$$T_{\mu,k} \to t_\mu \qquad (k\to\infty), \tag{52}$$

where t_μ is finite and positive, since $\left|\frac{\sigma-\lambda_\varkappa}{\sigma_\mu-\sigma}\right| \leq \frac{\delta}{d} < \frac{1}{2}$ $(\varkappa = 0, 1, \ldots)$. On the other hand we have

$$T_{\mu,k+1} - T_{\mu,k} = T_{\mu,k}\frac{\sigma-\lambda_{k+1}}{\sigma_\mu-\sigma} = O(\sigma-\lambda_{k+1})$$

and therefore further

$$\sum_{\varkappa=k}^{\infty}(T_{\mu,\varkappa+1}-T_{\mu,\varkappa}) = O(\sigma-\lambda_{k+1})\sum_{\varkappa=k+1}^{\infty}\left|\frac{\lambda_\varkappa-\sigma}{\lambda_{k+1}-\sigma}\right|.$$

But, by (48), the sum on the right is convergent and $< \frac{1}{1-\varepsilon}$; therefore, since the sum on the left is $t_\mu - T_{\mu,k}$, we obtain

$$T_{\mu,k} = t_\mu + O(\sigma-\lambda_{k+1}). \tag{53}$$

15. From this it follows further by (51) and (37) that

$$\frac{p_\mu}{M_{\mu,k}^2} = \frac{p_\mu}{t_\mu^2(\sigma_\mu-\sigma)^{2k+2}} + O\left(\frac{\lambda_{k+1}-\sigma}{d^{2k}}\right).$$

Therefore, if we put

$$p'_\mu = \frac{p_\mu}{t_\mu^2(\sigma_\mu-\sigma)},$$

we obtain finally for the numerator of D_k in (35) the expression

$$\sum_{\mu=1}^{m}{}' \frac{p'_\mu}{|\sigma_\mu-\sigma|^{2k}} + O\left(\frac{\lambda_{k+1}-\sigma}{d^{2k}}\right).$$

On the other hand the denominator in (35) can be written as $p+\eta_k$, where

$$\eta_k = \sum_{\mu=1}^{m}{}' p_\mu \prod_{\varkappa=0}^{k}\left(\frac{\sigma-\lambda_\varkappa}{\sigma_\mu-\lambda_\varkappa}\right)^2 \to 0 \qquad (k\to\infty),$$

since $\sigma - \lambda_\varkappa \to 0$ and $\sigma_\mu - \lambda_\varkappa \to \sigma_\mu - \sigma$. Therefore we now obtain

$$(54) \qquad (p + \eta_k)\, D_k = \sum_{\mu=1}^{m}{}' \frac{p'_\mu}{(\sigma_\mu - \sigma)^{2k}} + O\left(\frac{\lambda_{k+1} - \sigma}{d^{2k}}\right).$$

16. In the right hand sum in (54) it could happen that some values of $|\sigma_\mu - \sigma|$ occur twice, if there are two σ_μ symmetric with respect to σ, and it could even happen that two such terms cancel each other, if the corresponding p'_μ have the sum 0. Denote the *distinct* quotients $\frac{1}{(\sigma_\mu - \sigma)^2}$ which are not cancelled out by

$$(55) \qquad \gamma = \gamma_1 > \gamma_2 > \cdots > \gamma_r > 0.$$

Then we can write

$$(56) \qquad (p + \eta_k)\, D_k = \sum_{\varrho=1}^{r} s_\varrho \gamma_\varrho^k + O\left(\frac{\lambda_{k+1} - \sigma}{d^{2k}}\right),$$

where the s_ϱ are non-vanishing constants, as long as there are any terms left, that is if $r \geqq 1$.

17. But if we had $r = 0$, it would follow from (56) that

$$D_k = O\left(\frac{\lambda_{k+1} - \sigma}{d^{2k}}\right);$$

introducing this into (36) yields

$$\lambda_{k+1} - \sigma = O\left(\frac{\lambda_{k+1} - \sigma}{d^{2k}} M_k^2\right), \qquad d^{2k} = O(M_k^2),$$

and therefore by (47)

$$\left(\frac{d}{\delta}\right)^{2k} = O\left(\varepsilon^{2k(k-1)}\right),$$

which is impossible, since $0 < \varepsilon < 1$. Therefore we have $r \geqq 1$, and it follows from (56) and (48) that

$$(57) \qquad D_k \sim \frac{s_1}{p} \gamma^k \qquad (k \to \infty),$$

and (46) now follows from (49).

Thus far we have proved (46) only under the assumption that (45) holds. However, if we assume more generally that $\lambda_\varkappa \to \sigma$, for a certain $\varkappa_0$ we have $|\lambda_{\varkappa_0} - \sigma| \leqq \delta$, and our result above applies if we put $\lambda_{\varkappa + \varkappa_0} = \lambda'_\varkappa$. The theorem stated in Section 13 is now completely proved.

It is hardly necessary to add that our results hold also for Hermitian matrices, for which the discussion above remains valid with some slight and obvious modifications.

Note added October 1957. Professor G. Forsythe has directed my attention to a paper by S. H. Crandall, "Iterative procedures related to relaxation methods for eigenvalue problems" [Proc. Royal Soc. London, **207**, 416—423 (1951)], in which the iteration rules (3), (4) and (28), (29) are discussed. In particular, Professor Crandall establishes the *cubic character* of convergence of $\xi_\varkappa$ in the rule (28), (29). However he does not arrive at our asymptotic formula (46), which is the principal result of our paper.

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On a generalization of Airy's function

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1. The standard normal form (*cf.*, *e.g.*, [*7*], p. 97)

$$D^2 y = x^\mu y$$

($D = d/dx$) of the differential equation

$$t^2 Z'' + t Z' + (t^2 - \lambda) Z = 0$$

of the cylindrical functions $Z = Z_\lambda(t)$ of index λ, where $\mu = \mu(\lambda)$, can be reduced, if $\lambda = \frac{1}{3}$, $\lambda = -\frac{1}{3}$ or $\lambda = \frac{2}{3}$, to

(1) $$D^2 y - x y = 0;$$

cf. [*7*], p. 189, where (1) is written in the form

(2) $$D^2 y - \tfrac{1}{3} x y = 0$$

after a change of the unit of length on the x-axis. Since the work of Weyl [*8*], this particular differential equation has played a considerable part in the theory of singular Sturm-Liouville problems on a half-line $0 \leq x < \infty$ and, correspondingly, Kramers [*4*] observed that, to a good approximation, certain asymptotic problems concerning Schrödinger's equation (for an "arbitrary" potential) are determined by the asymptotic behavior of the same problem for (1) (concerning (1) itself, *cf.* the results of Zernike [*10*]).

What is relevant in these applications is a particular solution, $y(x) = A(x)$, of (2), this $y(x)$ being Airy's function

(3) $$y = A(x), \quad \text{where} \quad A(x) = \int_0^\infty \cos(u^3 - x u)\, du;$$

cf. [*7*], pp. 96—97. Since (3) is a solution of (2), it is clear from $D = d/dx$ that

(4) $$y = A(3^{-\frac{1}{3}} x)$$

is a solution of (1).

2. The purpose of this note is to point out how, by an appropriate application of an explicit rule, the relevant particular solution (3) of (1) can be extended so as to satisfy

(5) $$D^{m-1} y - x y = 0,$$

the generalization of (1) to an arbitrary $m \geqq 3$ (if $m=3$, then (5) reduces to (1); if $m=2$, then (5) becomes of first order and its general solution $y(x)$ is a constant multiple of e^X, where $X=-\frac{1}{2}x^2$).

The explicit result in question is very old and seems quite forgotten; it supplies the general solution of (5). The relevant particular solution of (5), to be obtained for any $m-1 \geqq 2$, will lead, if $m-1=2$, to an integral representation of $A(x)$ which is distinct from AIRY's integral (3).

3. JACOBI [*3*] wrote a note on an observation of SCHERK [*6*], which, with notations simplified so as to adapt them to the question at hand, can be formulated as follows: For every m (>1), the general solution (containing $m-1$ constants of integration $c_j:c_k$) of (5) is

$$(6) \qquad y(x)=\sum_{k=1}^{m} c_k \int_0^\infty \exp(\varepsilon^k x t - t^m/m)\,dt, \quad \text{where} \quad \sum_{k=1}^{m} c_k = 0,$$

if ε is a primitive m^{th} root of unity. The proof consists simply in successive partial integrations of the m definite integrals occurring in (6).

Disregard the trivial case $m=2$ of a first order equation (5), choose the two c_k's belonging to $\varepsilon^k=\exp(2\pi i/m)$ and $\varepsilon^k=1$ as $c_k=1$ and $c_k=-1$, respectively, and for the $m-2$ remaining c_k set $c_k=0$. Then the assumption $c_1+\cdots+c_m=0$ of (6) is satisfied, and (6) reduces to

$$(7) \quad y(x)=\int_0^\infty [\exp(\varepsilon x t)-\exp(x t)]\exp(-t^m/m)\,dt, \qquad \text{where} \quad \varepsilon=\exp(2\pi i/m).$$

But (5) is real for real x, and so both the real and imaginary parts of (7) must be solutions of (5). Since the real part of (7) differs from

$$(8) \qquad y(x)=\int_0^\infty [\exp(x t)-\exp(a_m x t)\cos(b_m x t)]\exp(-t^m/m)\,dt,$$

where

$$(9) \qquad a_m=\cos(2\pi/m), \qquad b_m=\sin(2\pi/m),$$

only by a constant factor if x is real, it follows that (8) is a solution of (5) even if x is complex.

Let x be confined to the half-line $0<x<\infty$, and let the integration variable t be replaced by xt with x fixed. Then the particular solution (8) of (5) appears in the form

$$(10) \qquad y(x)=x\int_0^\infty [\exp(x^2 t)-\exp(a_m x^2 t)\cos(b_m x^2 t)]\exp(-x^m t^m/m)\,dt.$$

Since the factor [ ] in (10) is majorized by a constant multiple of $\exp(x^2 t)$, it is clear from (10) that

$$(11) \qquad y(x)\to 0 \quad \text{as} \quad x\to\infty$$

(and that, as a matter of fact, $y(x)$ tends to 0 very fast indeed) if $m>2$. If, on the other hand, $m=2$, then (11) is clear from the explicit form of the solution of the differential equation (5) of order $m-1=1$, this solution being exponentially small for large x.

Needless to say, standard methods dealing with the asymptotic behavior of such integrals as (8) could readily replace (11) by explicit asymptotic formulae. But only (11) (or, for that matter, only the boundedness of (8) for large x) will be used later on.

4. The integral (8) is a particular solution of (5) whether m is even or odd. If m is restricted to be even, quite another real solution of (5) can be obtained.

It turns out that what then results is relevant for the cases $\alpha = 2, 4, 6, \ldots$ of the function

$$F_\alpha(x) = \frac{1}{\pi}\int_0^\infty \cos(xt)\exp(-t^\alpha)\,dt \tag{12}$$

(*cf.* [*9*], p. 834). The family $(0<\alpha<\infty)$ of these even functions $F_\alpha(x)$ was introduced by CAUCHY [*2*] as a generalization (from $\alpha = 2$ to an arbitrary, not necessarily integral, value of the positive parameter α) of LAPLACE'S symmetric normal distribution to an arbitrary symmetric "stable" distribution, with $F_\alpha(x)$ as the density of probability. *Cf.*, however, the discovery of F. BERNSTEIN [*1*], and the subsequent final results of P. LÉVY (*e.g.*, in [*5*], chap. V), concerning the *existence* of these "stable distributions".

If m is even, so that (5) becomes

$$D^{2n-1}y - xy = 0, \tag{13}$$

where $n = \frac{1}{2}m$, then both 1 and -1 are m-th roots of unity. In (6) choose as $\frac{1}{2}$ and $-\frac{1}{2}$, respectively, the values of the c_k belonging to $\varepsilon^k = 1$ and $\varepsilon^k = -1$, and for the $m-2$ remaining c_k set $c_k = 0$. Then the case $m = 2n$ of (6) reduces to

$$y(x) = \int_0^\infty \cosh(xt)\exp(-\tfrac{1}{2}t^{2n}/n)\,dt. \tag{14}$$

Hence (14) is a solution of (13). But it is clear that there exists a (unique) positive constant $\lambda = \lambda_{2n}$ for which the case $\alpha = 2n$ of CAUCHY'S transcendent (12) becomes identical with a constant multiple of (14) if x is replaced by $i\lambda x$ in (12).

Consequently, $y = F_{2n}(\lambda_{2n}x)$ *is a solution of*

$$(-1)^n D^{2n-1}y + xy = 0. \tag{13 bis}$$

5. Consider now (1), the case $m = 3$ of (5), so that (14), where $m = 2n$, is not applicable.

Since (9) shows that (8) now reduces to

$$y(x) = \int_0^\infty [\exp(xt) - \exp(-xt/2)\cos(3^{\frac{1}{2}}xt/2)]\exp(-t^3/3)\,dt, \tag{15}$$

(15) is a solution of (1). It will be shown that this solution of (1) is identical with a (positive) constant multiple of AIRY'S solution (4). In other words, if x is replaced by $3^{\frac{1}{3}}x$ in the integral (15), then the resulting function of x must be identical with $\gamma A(x)$, where $A(x)$ is AIRY'S integral (3) and γ a numerical constant (the value of which can be determined by placing $x = 0$ in both integrals).

This identification could be carried out by a deformation of the integration path $0 \leqq u < \infty$ of (3) in the complex u-plane. But this explicit work can be avoided if it is ascertained that (*i*) the solution (4) of (1) tends to 0 as $x \to \infty$,

and that (*ii*) *some* solution $y(x)$ of (1) must tend to ∞ as $x \to \infty$. In fact, (*ii*) implies that not all solutions $y(x)$ of (1) will satisfy (11), whereas (15) is a solution of (1) satisfying (11), as is, by (*i*), the solution (4) of (1). Thus the identification of (4) with a constant multiple of (15) can be concluded from (*i*) and (*ii*), since a homogeneous linear differential equation of second order does not have more than two solutions which are linearly independent.

Since the truth of (*i*) is exhibited by a well-known asymptotic formula of standard type (*cf.* [7], pp. 189—190 and pp. 202—203), only (*ii*) remains to be ascertained. But (*ii*) is quite on the surface; it follows from the fact that if (1) is written in the form

$$D^2 y = f(x)\, y, \tag{16}$$

then, since $f(x) = x$,

$$f(x) \geqq 0 \quad \text{for} \quad 0 \leqq x < \infty. \tag{17}$$

In fact, if $f(x)$ is any continuous function satisfying (17), and if $y(x)$ is that solution of (16) which is determined by the initial conditions $y(0) = 0$ and $D y(0) = 1$, then it is clear from (16) that, for reasons of convexity, $y(x) \geqq x$ will hold for $0 \leqq x < \infty$, and so $y(x) \to \infty$ as $x \to \infty$.

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Boundedness of Solutions of Linear Differential Systems with Periodic Coefficients

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In previous papers L. CESARI, J. K. HALE, and R. A. GAMBILL [*1, 3, 7, 8, 13, 15, 17*] have studied classes of systems of linear differential equations, all reducible to the form

$$x' = A\,x + \varepsilon\,\Phi(t)\,x, \tag{1}$$

where $x = \mathrm{col}(x_1, \ldots, x_n)$, ε is a small parameter, $A = [a_{jh}]$ an $n\times n$ constant matrix, $\Phi(t) = [\varphi_{jh}(t)]$ an $n\times n$ matrix whose elements are periodic functions of t of given period $T = 2\pi/\omega$, L-integrable in $[0, T]$. Theorems of boundedness and criteria for unboundedness of the solutions of (1) have been given in [*1, 3, 7, 8, 13, 15*] in case the characteristic roots of A are all $\neq 0$, distinct, and purely imaginary. In [*17*] it is assumed that A has also a number of zero characteristic roots. In all cases it was shown that convenient conditions of symmetry, or of evenness and oddness of the periodic coefficients, may assure boundedness.

In the present paper it is assumed that some of the characteristic roots of A are distinct and purely imaginary and one possibly zero, and the remaining characteristic roots are real or complex with real parts negative. The coefficients in ε^2 of the Floquet characteristic exponents of system (1) are given in explicit and easily computable form. Consequently simple criteria of boundedness and unboundedness follow (§ 3). Numerous examples are given showing the critical part that such coefficients have in the present essentially more general situation. The same method of successive approximations (§ 2) is used in the present paper as in the ones quoted above, namely, the method introduced by L. CESARI in [*1*] and successively developed by L. CESARI, J. K. HALE, R. A. GAMBILL, W. R. FULLER for proving theorems of boundedness of linear differential systems with periodic coefficients *(loc. cit.)* and existence theorems for cycles of weakly non-linear differential systems (periodic or autonomous) [*2, 4, 9, 14, 16*]. Nevertheless some of the theorems of § 3 could be obtained also by a different method.

§ 1. Preliminary remarks

1.1. Characteristic exponents. We shall first recall some properties of linear differential systems with periodic coefficients, due essentially to G. FLOQUET [*5, 6, 10*]. Let us consider the system

$$x' = P(t)\,x, \tag{1.1.1}$$

where $x = \operatorname{col}(x_1, \ldots, x_n)$, $P(t)$ is an $n \times n$ matrix whose elements $p_{jh}(t)$ are complex-valued functions of the real variable t, periodic of period $T = 2\pi/\omega$, L-integrable in $[0, T]$. We shall consider only those solutions $x(t) = [x_1(t), \ldots, x_n(t)]$ of (1.1.1) whose elements $x_j(t)$ are absolutely continuous (AC) functions of t and which satisfy (1.1.1) almost everywhere (a.e.). Then there exists a fundamental system of n AC solutions of (1.1.1)

$$x^{(h)}(t) = [x_{jh}(t),\ j = 1, \ldots, n], \qquad h = 1, \ldots, n,$$

which can be distributed in disjoint groups (HAMBURGER'S groups) $x^{(h)}(t)$, $h = k,\ k+1, \ldots, k+\lambda-1$, $1 \leq k \leq k+\lambda-1 \leq n$, each of which satisfies the following relations:

$$x^{(k)}(t+T) = s\, x^{(k)}(t), \qquad x^{(k)}(t) = e^{\alpha t} z^{(k,0)}(t), \qquad s = e^{\alpha T},$$

$$x^{(h)}(t+T) = s\, x^{(h)}(t) + x^{(h-1)}\, t,$$

$$x^{(h)}(t) = e^{\alpha t}[z^{(h,0)}(t) + t\, z^{(h,1)}(t) + \cdots + t^{h-1} z^{(h,h-k)}(t)], \quad h = k, \ldots, k+\lambda-1,$$

where s, α are complex numbers (a pair for each group) and all vectors z have elements which are AC periodic functions of period T [$z^{(h,h-k)}(t)$ non-identically zero, $h = k, \ldots, k+\lambda-1$].

The numbers s above are the characteristic multipliers and the numbers α (defined mod ωi) are the characteristic exponents. If s_l, $l = 1, \ldots, m$, are all the distinct characteristic multipliers, and if α_l are the corresponding characteristic exponents, if μ_l is the sum of all the numbers λ above relative to all the Hamburger groups having the same $s = s_l$, then $\mu_1 + \cdots + \mu_n = n$. The following further information on the characteristic multipliers is needed. Consider the fundamental system $X^{(h)}(t) = [X_{jh}(t), j = 1, \ldots, n]$, $h = 1, \ldots, n$, of solutions of (1.1.1) verifying the initial conditions $X_{jh}(0) = \delta_{jh} = 1$ if $j = h$, $= 0$ if $j \neq h$. Then the characteristic roots of the matrix $[X_{jh}(T)]$ are the numbers $s_1, \ldots, s_m$ with multiplicities $\mu_1, \ldots, \mu_m$. We shall denote by $s_1, \ldots, s_n$ and $\alpha_1, \ldots, \alpha_n$ the characteristic multipliers and exponents, each repeated as many times as the multiplicity μ of s.

From the considerations above it follows that the solutions of (1.1.1) are bounded in $[0, +\infty)$ if and only if all characteristic exponents α have non-positive real parts, and those with zero real parts correspond to HAMBURGER'S groups all made up of only one element ($\lambda = 1$). A sufficient condition (though not necessary) is, therefore, that all characteristic exponents α have non-positive real parts and those with zero real parts correspond to characteristic multipliers s_l which are simple ($\mu_l = 1$). The case where P is a constant matrix in t is no exception. Then T is arbitrary, and we can assume that the characteristic exponents α_l, $l = 1, 2, \ldots, n$, are the characteristic roots of the matrix P.

Let us consider now a system

$$x' = Q(t, \varepsilon)\, x, \tag{1.1.2}$$

where $x = \operatorname{col}(x_1, \ldots, x_n)$, $Q(t, \varepsilon) = [q_{jh}(t, \varepsilon)]$ is an $n \times n$ matrix such that (*a*) all $q_{jh}(t, \varepsilon)$ are periodic functions of t of period T, L-integrable in $[0, T]$ for every $|\varepsilon| < \varepsilon_0$ and some $\varepsilon_0 > 0$; (*b*) all $q_{jh}(t, \varepsilon)$ are holomorphic in ε for $|\varepsilon| < \varepsilon_0$ for every t; (*c*) $|q_{jh}(t, \varepsilon)| \leq \psi(t)$ for all j, h, t, $|\varepsilon| < \varepsilon_0$, where $\psi(t)$ is a non-negative periodic function of t of period T, L-integrable in $[0, T]$. Then the solutions $X_{jh}(t, \varepsilon)$,

considered above, are holomorphic functions of ε for $|\varepsilon|<\varepsilon_0$ and for every t [*cf.*, *e.g.*, [5], Chapters 1 and 2]. Thus it follows that the multipliers s_l, $l=1, \ldots, n$ (*i.e.*, the characteristic roots of $[X_{jh}(T, \varepsilon)]$), are the roots of an algebraic equation $s^n+a_1(\varepsilon)\, s^{n-1}+\cdots+a_n(\varepsilon)=0$, whose coefficients are holomorphic functions of ε for $|\varepsilon|<\varepsilon_0$ and $a_n(\varepsilon)=\det[X_{jh}]\neq 0$. Hence the multipliers s_h can be thought of as analytic functions of ε with at most branch points of finite order for $|\varepsilon|<\varepsilon_0$, and also we have $s_l\neq 0$, $l=1, \ldots, n$. Finally the characteristic exponents $\alpha_h(\varepsilon)=T^{-1}\log s_l(\varepsilon)$, $l=1, \ldots, n$, can be thought of as analytic functions of ε for $|\varepsilon|<\varepsilon_0$, with at most branch points of finite order, once the values $\alpha_l(0)$, $l=1, \ldots, n$, have been chosen. For instance, for the system $x_1'=x_1+\varepsilon x_2$, $x_2'=x_1+x_2$, $T>0$ arbitrary, we may assume $\alpha_1(\varepsilon)$, $\alpha_2(\varepsilon)=1\pm\sqrt{\varepsilon}$.

Analogously, for a system as (1.1.2), where $\varepsilon=(\varepsilon_1, \ldots, \varepsilon_\mu)$ is a vector parameter and the above conditions of holomorphism of Q hold for $\varepsilon_1, \ldots, \varepsilon_\mu$ in some region K of the $(\varepsilon_1, \ldots, \varepsilon_\mu)$ complex space, the analogous conclusion holds concerning the analyticity of the characteristic exponents $\alpha_h(\varepsilon)$ for ε in K.

1.2. The concept of mean value. Let C_ω denote the family of all functions which are finite sums of functions of the form $f(t)=e^{\alpha t}\varphi(t)$, $-\infty<t<+\infty$, where α is any complex number and $\varphi(t)$ is any complex-valued function of the real variable t, periodic of a given period T, L-integrable in $[0, T]$. Obviously a function of the form $f=e^{\alpha t}\varphi(t)$ admits the infinitely many decompositions of the same type which are obtained by replacing α and φ by $\alpha+iK\omega$ and $\varphi e^{-iK\omega t}$, $K=0, \pm 1, \ldots$. If $\varphi(t)$ has the Fourier series

$$\varphi(t)\sim\sum_{n=-\infty}^{+\infty} c_n e^{in\omega t},$$

then we shall denote the series

(1.2.1) $$f(t)=e^{\alpha t}\varphi(t)\approx\sum_{n=-\infty}^{+\infty} c_n e^{in\omega t+\alpha t}$$

as the series associated with $f(t)$. Following L. Cesari [1], we shall denote by mean value, $m[f]$ of $f(t)$, the number $m[f]=0$ if $in\omega+\alpha\neq 0$ for all n, $m[f]=c_n$ if $in\omega+\alpha=0$ for some n. Both the series (1.2.1) and $m[f]$ are independent of the chosen decomposition of $e^{\alpha t}\varphi(t)$. Finally let us define $m[f]$ in the class C_ω as an additive functional (for more details, see L. Cesari [1] and J. K. Hale [12]). Obviously $m[f]$ reduces to the usual mean value for periodic functions $f(t)$ of period T, and, in this case, we have $m[f]=T^{-1}\int_0^T f(t)\,dt$.

The following statements are needed in the sequel.

(1.2.i) *If $f(t)\in C_\omega$ and $f(t)=e^{(a+ib)t}g(t)$, a, b real, $g(t)$ periodic of period T, and $m[f]=0$, then there is a unique primitive of $f(t)$, say $F(t)$, which belongs to C_ω, and such that $m[F]=0$. Its associated series is obtained by formal integration of the series associated with $f(t)$* (L. Cesari [1], J. K. Hale [12]).

(1.2.ii) *Under the conditions of* (1.2.i) *we have*

$$|F(t)|\leqq N\int_0^T|g(t)|\,dt, \qquad N=N(a+ib, T), \qquad 0\leqq t\leqq T,$$

where N is a constant depending only on a, b, T and not on g (J. K. Hale [12]).

The proof of (1.2.ii) given in [12] is based on the use of convolution integrals. The particular case $a=b=0$ is of interest. Here $N(0, 0, T)=2^{-1}$, as can be proved directly as follows. First suppose $f=g$, real. Let us observe that $\int_0^T f(t)\,dt=0$, $\int_0^T F(t)\,dt=0$, and that it is not restrictive to suppose that the continuous function F is not constant and takes, at $t=0$ and $t=\xi$, $0<\xi<T$, respectively, its absolute minimum F_1 and its absolute maximum F_2, $F_1<0<F_2$. Then

$$0<F_2<F_2-F_1=\int_0^\xi f\,dt, \qquad 0>F_1>F_1-F_2=\int_\xi^T f\,dt,$$
$$0<2(F_2-F_1)=\left(\int_0^\xi-\int_\xi^T\right)f\,dt\leqq\int_0^T|f|\,dt=M.$$

Thus $|F|\leqq F_2-F_1\leqq 2^{-1}M$. If f, F are complex, and $f=\varphi_1+i\varphi_2$, $F=\Phi_1+i\Phi_2$, $\varphi_1, \varphi_2, \Phi_1, \Phi_2$ real, then $|\Phi_j|\leqq 2^{-1}\int_0^T|\varphi_j|\,dt$, $j=1, 2$, and

$$|F|=(\Phi_1^2+\Phi_2^2)^{\frac{1}{2}}\leqq 2^{-1}\left[\left(\int_0^T|\varphi_1|\,dt\right)^2+\left(\int_0^T|\varphi_2|\,dt\right)^2\right]^{\frac{1}{2}}\leqq 2^{-1}\int_0^T(\varphi_1^2+\varphi_2^2)^{\frac{1}{2}}\,dt.$$

Thus $|F|\leqq 2^{-1}\int_0^T|f|\,dt$. This proves that we may assume $N(0, 0, T)=2^{-1}$. This is also the best evaluation. Indeed, if $f=1$ for $0\leqq t<2^{-1}a$, $f=-1$ for $2^{-1}a\leqq t<a$, $f=-m$ for $a\leqq t<2^{-1}(1+a)$, $f=m$ for $2^{-1}(1+a)\leqq t<1$, where $0<a<2^{-1}$, $m=a^2(1-a)^{-2}$, and $T=1$, then $F=t,\ a-t,\ -m(t-a),\ m(t-1)$, respectively, in the four intervals above, and $|F|_{\max}=2^{-1}a$, $\int_0^1|f|\,dt=a(1-a)^{-1}$. Thus $|F|_{\max}:\int_0^1|f|\,dt=2^{-1}(1-a)\to 2^{-1}$ as $a\to 0+$.

§ 2. Expressions for the solutions and the characteristic exponents

2.1. Notations and hypotheses. Let us consider the linear system of differential equations

$$y'=Ay+\varepsilon\Phi(t)\,y, \tag{2.1.1}$$

where $y=\mathrm{col}(y_1, \ldots, y_n)$, ε a parameter, $A=[a_{jh}]$ a constant matrix, Φ an $n\times n$ matrix whose elements $\varphi_{jh}(t)$ are complex-valued functions, periodic of period $T=2\pi/\omega$, L-integrable in $[0, T]$. We shall suppose that of the n characteristic roots ϱ_j, $j=1, \ldots, n$, of A, k roots, say ϱ_j, $j=1, \ldots, k$, $0\leqq k\leqq n$, are distinct, have real parts $R[\varrho_j]=0$, with $\varrho_j\not\equiv\varrho_h \pmod{\omega i}$, and the remaining $n-k$, say ϱ_j, $j=k+1, \ldots, n$, have $R[\varrho_j]<0$ and are not necessarily distinct. If we denote by $\varrho_1', \ldots, \varrho_m'$, the m distinct values of the n characteristic roots $\varrho_1, \ldots, \varrho_n$, and $\mu_1, \ldots, \mu_m$ their multiplicities, we have $0\leqq k\leqq m\leqq n$, $\mu_1=\cdots=\mu_k=1$, $\mu_{k+1}, \ldots, \mu_m\geqq 1$, $\mu_1+\cdots+\mu_m=n$.

It is not restrictive to take $A=[a_{jh}]$ in canonical form, that is, $A=\mathrm{diag}(A_1, A_2, \ldots, A_m)$, with $A_j=[\varrho_j]$, $j=1, \ldots, k$, $A_s=[a_{jh}]$, $s=k+1, \ldots, m$, where $k'+1\leqq j,\ h\leqq k'+\mu_s$, $k'=\mu_1+\cdots+\mu_{s-1}$, and $a_{jj}=\varrho_j=\varrho_s'$, $a_{j+1,j}=1$, or $=0$, $a_{jh}=0$ for $j-h\neq 0, 1$, $k'+1\leqq j$, $h\leqq k'+\mu_s$. If $\delta_{j+1}=a_{j+1,j}$, then $\delta_j=1$ or 0, for $j=k+1, \ldots, n$, and $\delta_j=0$ for $j=2, \ldots, k$.

Let $\alpha_1(\varepsilon), \ldots, \alpha_n(\varepsilon)$ be the characteristic exponents of (2.1.1). For $\varepsilon = 0$ system (2.1.1) has constant coefficients and, by § 1, we may suppose $\alpha_j(0) = \varrho_j$, $j = 1, \ldots, n$. By § 1 we know that the functions $\alpha_j(\varepsilon)$ are analytic functions of ε with at most branch points of finite order (for ε complex and finite).

If we take in the complex plane arbitrary disjoint neighborhoods U_j of the m distinct points ϱ_j', $j = 1, \ldots, m$, then for $|\varepsilon|$ sufficiently small, each point $\alpha_j(\varepsilon)$ $j = 1, \ldots, k$, is in the corresponding U_j, and the remaining $n-k$ points $\alpha_j(\varepsilon)$ (distinct or not) are distributed in groups of μ_j in each U_j, $j = k+1, \ldots, m$. Thus for $|\varepsilon|$ sufficiently small, say for $|\varepsilon| < \varepsilon_0$ and some $\varepsilon_0 > 0$, we certainly have

$$\alpha_j(\varepsilon) \not\equiv \alpha_h(\varepsilon) \pmod{\omega i}, \qquad j \neq h, \quad j, h = 1, \ldots, k,$$

$$R[\alpha_j(\varepsilon)] < 0, \qquad j = k+1, \ldots, n.$$

This implies that the question of the boundedness in $[0, +\infty)$ of the solutions of (2.1.1) for $|\varepsilon| < \varepsilon_0$ is reduced to the determination of the signs of $R[\alpha_j(\varepsilon)]$, $j = 1, \ldots, k$. To achieve this we shall determine k solutions of (2.1.1) of the form $x_j = e^{\alpha_j(\varepsilon)t} z_j(t, \varepsilon)$, where $z_j(t, \varepsilon)$ denote periodic vector functions of period $T = 2\pi/\omega$ in t. To obtain these k solutions we shall use the casting out method of successive approximations mentioned in the introduction.

2.2. A method of successive approximations. Let us consider system (2.1.1) where A has the canonical form described above. Let us consider the auxiliary system

$$y' = By + \varepsilon \Phi(t) y, \tag{2.2.1}$$

where the matrix B is identical with A except that the letters $\varrho_1, \ldots, \varrho_n$ are replaced by $\tau_1, \ldots, \tau_n$. In harmony with a remark made in [*1*] we shall actually assume

$$\tau_j = \varrho_j \quad \text{for all} \quad j = 1, \ldots, n, \quad j \neq h, \tag{2.2.2}$$

where h is fixed, $1 \leq h \leq k$. Since $\varrho_j \not\equiv \varrho_h \pmod{\omega i}$, $j = 1, \ldots, k$, $j \neq h$, there is a circle γ_h of center ϱ_h in the complex plane such that $\tau_h \not\equiv \varrho_j \pmod{\omega i}$, $j \neq h$, $j = 1, \ldots, k$, for every $\tau_h \in \gamma_h$. Since $R(\varrho_j) < 0$ for $j = k+1, \ldots, n$, we can take γ_h sufficiently small that $R(\varrho_j) < R(\tau_h)$ for all $\tau_h \in \gamma_h$, $j = k+1, \ldots, n$. Hence we have $\tau_j \not\equiv \tau_h \pmod{\omega i}$ for all $j = 1, \ldots, n$, $j \neq h$. We shall consider τ_h as an indeterminate, $\tau_h \in \gamma_h$.

By the method described below we shall obtain a solution $y_h = (y_{1h}, \ldots, y_{nh})$ of a system

$$y' = Cy + \varepsilon \Phi(t) y, \tag{2.2.3}$$

where C is identical with B except that τ_h is replaced by $\tau_h - \varepsilon d_h$ and d_h is a convenient holomorphic function of ε. In order that the solution of (2.2.3) be a solution of (2.1.1) we must require that $\tau_h - \varepsilon d_h = \varrho_h$, $\tau_h \in \gamma_h$, having already assumed that $\tau_j = \varrho_j$ for all $j \neq h$, $j = 1, \ldots, n$.

If we replace y_{jh} in (2.2.1) by

$$y_{jh} = x_{jh}^{(0)} + \varepsilon x_{jh}^{(1)} + \varepsilon^2 x_{jh}^{(2)} + \cdots, \qquad j = 1, \ldots, n, \tag{2.2.4}$$

where h is a fixed integer, $1 \leqq h \leqq k$, and equate like powers of ε we obtain

$$\frac{d x_{jh}^{(0)}}{dt} = \tau_j x_{jh}^{(0)}, \qquad j = 1, \ldots, k+1,$$

$$\frac{d x_{jh}^{(0)}}{dt} = \delta_{j-1} x_{j-1,h}^{(0)} + \tau_j x_{jh}^{(0)}, \qquad j = k+2, \ldots, n,$$

$$\frac{d x_{jh}^{(m)}}{dt} = \tau_j x_{jh}^{(m)} + \sum_{r=1}^{n} \varphi_{jr} x_{rh}^{(m-1)}, \qquad j = 1, \ldots, k+1, \quad m \geqq 1,$$

$$\frac{d x_{jh}^{(m)}}{dt} = \delta_{j-1} x_{j-1,h}^{(m)} + \tau_j x_{jh}^{(m)} + \sum_{r=1}^{n} \varphi_{jr} x_{rh}^{(m-1)}, \qquad j = k+1, \ldots, n, \quad m \geqq 1,$$

whose solutions satisfying the condition $x_{jh}^{(0)}(0) = \delta_{jh}$, $x_{jh}^{(m)}(0) = 0$, $m \geqq 1$, where as usual $\delta_{hh} = 1$, $\delta_{jh} = 0$ for $j \neq h$, $j = 1, \ldots, n$, are given by the expressions

$$x_{jh}^{(0)} = \delta_{jh} e^{\tau_h t}, \qquad j = 1, \ldots, n,$$

$$x_{jh}^{(m)} = e^{\tau_j t} \int_0^t e^{-\tau_j \alpha} \sum_{r=1}^{n} \varphi_{jr} x_{rh}^{(m-1)} \, d\alpha, \qquad j = 1, \ldots, k+1,$$

$$x_{jh}^{(m)} = e^{\tau_j t} \int_0^t e^{-\tau_j \alpha} \left[\delta_{j-1} x_{j-1,h}^{(m)} + \sum_{r=1}^{n} \varphi_{jr} x_{rh}^{(m-1)}\right] d\alpha, \qquad j = k+2, \ldots, n, \quad m = 1, 2, \ldots.$$

As is apparent, the functions successively obtained by this process are not of class C_ω (in general), since it soon becomes necessary to integrate functions of class C_ω whose mean values are $\neq 0$ [*cf.* Theorem (1.2.i)]. Successively, polynomial terms in t occur of higher and higher degree. According to the general idea of the method as given in [*1, 4, 9, 13, 14*] the process above has to be modified by subtracting the mean value (at least) from each integrand and by choosing at each integration the unique primitive of class C_ω and mean value zero in terms of Theorem (1.2.i). By a convenient formalization the new functions $x_{jh}^{(m)}$ so obtained will satisfy modified differential equations and modified initial conditions, and the corresponding functions (2.2.4) will satisfy (2.2.3), as will be shown.

According to one of the equivalent formalizations of the method, namely the one given in [*9, 14*], let us put

$$S_{lh} = m\left[e^{-\tau_h t} \sum_{j=1}^{n} \varphi_{hj} x_{jh}^{(l-1)}\right], \qquad l = 1, 2, \ldots, \tag{2.2.5}$$

and define the successive approximations by

$$\begin{aligned}
x_{jh}^{(0)} &= \delta_{jh} e^{\tau_h t}, \qquad j = 1, \ldots, n, \\
x_{hh}^{(m)} &= e^{\tau_h t} \int e^{-\tau_h t} \left[\sum_{r=1}^{n} \varphi_{hr} x_{rh}^{(m-1)} - \sum_{l=1}^{m} S_{lh} x_{hh}^{(m-l)}\right] dt, \qquad m \geqq 1, \\
x_{jh}^{(m)} &= e^{\tau_j t} \int e^{-\tau_j t} \sum_{r=1}^{n} \varphi_{jr} x_{rh}^{(m-1)} \, dt, \qquad j \neq h, \quad j = 1, 2, \ldots, k+1, \quad m \geqq 1, \\
x_{jh}^{(m)} &= e^{\tau_j t} \int e^{-\tau_j t} \left[\sum_{r=1}^{n} \varphi_{jr} x_{rh}^{(m-1)} + \delta_{j-1} x_{j-1,h}^{(m)}\right] dt, \quad j = k+2, \ldots, n, \quad m \geqq 1,
\end{aligned} \tag{2.2.6}$$

where the primitives are the unique primitives of (1.2.i). We will show that the integrands of (2.2.6) have mean values zero.

We first show by induction that for $m > 0$ we have

$$x_{jh}^{(m)} = e^{\tau_h t}\, \psi_{jh}^{(m)}(t), \qquad j = 1, 2, \ldots, n,$$

where $\psi_{jh}^{(m)}(t)$ are periodic of period T and $m[\psi_{hh}^{(m)}] = 0$. For $m = 1$ we have

$$x_{hh}^{(1)} = e^{\tau_h t} \int e^{-\tau_h t} [\varphi_{hh}\, e^{\tau_h t} - m[\varphi_{hh}]\, e^{\tau_h t}]\, dt = e^{\tau_h t}\, \psi_{hh}^{(1)}(t),$$

where $\psi_{hh}^{(1)}$ is the unique primitive of mean value zero. We have also

$$x_{jh}^{(1)} = e^{\tau_j t} \int e^{-\tau_j t}\, \varphi_{jh}\, e^{\tau_h t}\, dt = e^{\tau_h t}\, \psi_{jh}^{(1)}(t), \qquad j \neq h, \quad j = 1, 2, \ldots, k+1,$$

$$x_{jh}^{(1)} = e^{\tau_j t} \int e^{-\tau_j t} [\varphi_{jh}\, e^{\tau_h t} + \delta_{j-1}\, e^{\tau_h t}\, \psi_{j-1,h}^{(1)}]\, dt = e^{\tau_h t}\, \psi_{jh}^{(1)}(t), \qquad j = k+2, \ldots, n,$$

where $\psi_{jh}^{(1)}$, $j = 1, \ldots, n$, are periodic of period T. The integrands have mean value zero (see § 1.2) since $\tau_j \not\equiv \tau_h \pmod{\omega i}$, $j \neq h$, $j = 1, \ldots, n$. We complete the induction by assuming $x_{jh}^{(l)} = e^{\tau_h t}\, \psi_{jh}^{(l)}(t)$, $j = 1, \ldots, n$, $l = 1, \ldots, m-1$, where the $\psi_{jh}^{(l)}$ are periodic of period T and $m[\psi_{hh}^{(l)}] = 0$. If we replace the $x_{jh}^{(l)}$, $l = 1, \ldots, m-1$, in (2.2.6) by the expressions above, we obtain

$$x_{hh}^{(m)} = e^{\tau_h t} \int \left[\sum_{r=1}^{n} \varphi_{hr}\, \psi_{rh}^{(m-1)} - S_{mh} - \sum_{l=1}^{m-1} S_{lh}\, \psi_{hh}^{(m-l)} \right] dt = e^{\tau_h t}\, \psi_{hh}^{(m)}(t)$$

where the $\psi_{hh}^{(m)}(t)$ are the unique primitives of mean value zero. The integrand has mean value zero since

$$S_{mh} = m\left[\sum_{r=1}^{n} \varphi_{hr}\, \psi_{rh}^{(m-1)} \right] \quad \text{and} \quad m[\psi_{hh}^{(m-l)}] = 0, \qquad l = 1, \ldots, m-1.$$

We have also

$$x_{jh}^{(m)} = e^{\tau_j t} \int e^{(\tau_h - \tau_j)t} \sum_{r=1}^{n} \varphi_{jr}\, \psi_{rh}^{(m-1)}\, dt = e^{\tau_h t}\, \psi_{jh}^{(m)}(t), \qquad j \neq h, \quad j = 1, \ldots, k+1,$$

$$x_{jh}^{(m)} = e^{\tau_j t} \int e^{(\tau_h - \tau_j)t} \left[\sum_{r=1}^{n} \varphi_{jr}\, \psi_{rh}^{(m-1)} + \delta_{j-1}\, \psi_{j-1,h}^{(m)} \right] dt = e^{\tau_h t}\, \psi_{jh}^{(m)}(t), \qquad j = k+2, \ldots, n,$$

where the $\psi_{jh}^{(m)}(t)$ are periodic of periodic T. The integrands have mean value zero since $\tau_j \not\equiv \tau_h \pmod{\omega i}$, $j \neq h$, $j = 1, \ldots, n$. The induction is complete, and we can write

$$y_{jh} = \sum_{m=0}^{\infty} \varepsilon^m\, x_{jh}^{(m)} = e^{\tau_h t} \sum_{m=0}^{\infty} \varepsilon^m\, \psi_{jh}^{(m)}, \qquad j = 1, \ldots, n. \tag{2.2.7}$$

Remark. The formalization of the method described here follows closely the one given in [*9*] and [*14*] for nonlinear systems. The proof of its convergence given below is a somewhat shorter form of the one given in [*9*] and [*14*]. Details are pointed out which are needed in § 3. For a formalization valid for linear and nonlinear problems under the Lipschitz condition alone, see [*2*].

2.3. Proof of convergence. We now show that series (2.2.7) is absolutely and uniformly convergent for $|\varepsilon|$ sufficiently small and for all t by determining majorants ξ_m such that

$$|\psi_{jh}^{(m)}(t)| \leq \xi_m, \qquad j = 1, \ldots, n, \; m = 0, 1, \ldots, \tag{2.3.1}$$

for all t, and such that $\sum_{m=0}^{\infty} \xi_m \varepsilon^m$ is absolutely convergent for $|\varepsilon|$ sufficiently small. We can choose $\xi_0 = 1$ since $\psi_{hh}^{(0)} = 1$ and $\psi_{jh}^{(0)} = 0$, $j \neq h$. Let us assume that we have the first $m-1$ majorants, that is, $|\psi_{jh}^{(l)}| \leqq \xi_l$, $j = 1, \ldots, n$, $l = 1, 2, \ldots, m-1$. Then we have

$$
\begin{aligned}
|S_{lh}| &= \left| m \left[e^{-\tau_h t} \sum_{j=1}^{n} \varphi_{hj} x_{jh}^{(l-1)} \right] \right| = \left| T^{-1} \int_0^T \sum_{j=1}^{n} \varphi_{hj} \psi_{jh}^{(l-1)} \, dt \right| \\
&\leqq T^{-1} \int_0^T \sum_{j=1}^{n} |\varphi_{hj}| \, |\psi_{jh}^{(l-1)}| \, dt \leqq n \, \xi_{l-1} M T^{-1}, \qquad l = 1, \ldots, m,
\end{aligned} \tag{2.3.2}
$$

where

$$
M = \max \int_0^T |\varphi_{jr}(t)| \, dt, \qquad j, r = 1, \ldots, n,
$$

and finally, by (1.2),

$$
\begin{aligned}
|\psi_{hh}^{(m)}(t)| &= \left| \int \left[\sum_{r=1}^{n} \varphi_{hr} \psi_{rh}^{(m-1)} - \sum_{l=1}^{m} S_{lh} \psi_{hh}^{(m-l)} \right] dt \right| \\
&\leqq n \, \xi_{m-1} M + \sum_{l=1}^{m} (n \, \xi_{l-1} M T^{-1}) \, \xi_{m-l} T, \qquad 0 \leqq t \leqq T.
\end{aligned}
$$

We have also, by (1.2),

$$
\begin{aligned}
|\psi_{jh}^{(m)}(t)| &= \left| e^{(\tau_j - \tau_h) t} \int e^{-(\tau_j - \tau_h) t} \sum_{r=1}^{n} \varphi_{jr} \psi_{rh}^{(m-1)} \, dt \right| \\
&\leqq e^{|R(\tau_j - \tau_h) T|} N_{jh} \, n \, M \, \xi_{m-1} = K_{jh} \, n \, M \, \xi_{m-1},
\end{aligned}
$$

for all $0 \leqq t \leqq T$, $j = 1, \ldots, k+1$, $j \neq h$, where N_{jh} are the constants $N(\tau_h - \tau_j, T)$ of (1.2.ii), and $K_{jh} = N_{jh} e^{|R(\tau_j - \tau_h)|}$. Successively we have

$$
\begin{aligned}
|\psi_{k+2,h}^{(m)}(t)| &= \left| e^{(\tau_{k+2} - \tau_h) t} \int e^{-(\tau_{k+2} - \tau_h) t} \left[\sum_{r=1}^{n} \varphi_{k+2,r} \psi_{rh}^{(m-1)} + \delta_{k+1} \psi_{k+1,h}^{(m)} \right] dt \right| \\
&\leqq e^{|R(\tau_{k+2} - \tau_h)|} \left[N_{k+2,h} (n \, M \, \xi_{m-1} + \delta_{k+1} K_{k+1,h} T \, n \, M \, \xi_{m-1}) \right] \\
&= n \, M \, \xi_{m-1} K_{k+2,h}, \qquad 0 \leqq t \leqq T,
\end{aligned}
$$

where $N_{k+2,h}$ is the constant $N(\tau_h - \tau_{k+2}, T)$ of (1.2.ii), and

$$
K_{k+2,h} = N_{k+2,h} (1 + \delta_{k+1} K_{k+1,h} T) \, e^{|R(\tau_{k+2} - \tau_h)|}.
$$

Analogously we have

$$
|\psi_{jh}^{(m)}(t)| \leqq n \, M \, \xi_{m-1} K_{jh}, \qquad j = k+3, \ldots, n,
$$

for convenient constants K_{jh} and all $0 \leqq t \leqq T$. If we choose $P = \max(n M K_{jh}, n M, 1)$ for all $j = 1, \ldots, n$, then we have

$$
|\psi_{jh}^{(m)}(t)| \leqq P \left(\xi_{m-1} + \sum_{l=1}^{m} \xi_{l-1} \xi_{m-l} \right)
$$

for all $0 \leqq t \leqq T$ and $j = 1, 2, \ldots, n$. Thus we can choose

$$
\xi_m = P \left(\xi_{m-1} + \sum_{l=1}^{m} \xi_{l-1} \xi_{m-l} \right), \tag{2.3.3}
$$

and we have $|\psi_{jh}^{(m)}(t)| \leq \xi_m$ for all $0 \leq t \leq T$. Relation (2.3.3) defines ξ_m for all $m=1, 2, \ldots$, and thus the inequalities (2.3.1) are proved for all $0 \leq t \leq T$. Since the functions $\psi_{jh}^{(m)}(t)$ are all periodic of period T the same inequalities are proved for all t.

It remains to show that $\sum_{m=0}^{\infty} \xi_m |\varepsilon|^m$ converges for $|\varepsilon|$ sufficiently small. Let us consider the function

$$F(\xi, \varepsilon) = \varepsilon P \xi^2 + \varepsilon P \xi - (\xi - \xi_0), \tag{2.3.4}$$

ε, ξ real, and observe that $F(\xi_0, 0) = 0$, $F_\xi(\xi_0, 0) = -1$. Thus by the implicit function theorem, we conclude that there exists a real function $\xi = \xi(\varepsilon)$ satisfying $F(\xi, \varepsilon) = 0$, $\xi_0 = \xi(0)$, for all $|\varepsilon| \leq \varepsilon_0$ and some $\varepsilon_0 > 0$ sufficiently small. We have also, for ε_0 sufficiently small and all $|\varepsilon| \leq \varepsilon_0$,

$$\xi = \xi(\varepsilon) = \sum_{m=0}^{\infty} a_m \varepsilon^m, \tag{2.3.5}$$

where this series is absolutely convergent for all $|\varepsilon| \leq \varepsilon_0$. If we replace $\xi(\varepsilon)$ in (2.3.4) by its expansion (2.3.5) and equate like powers of ε, we obtain

$$a_0 = \xi_0, \qquad a_m = P a_{m-1} + P \sum_{k=1}^{m} a_{k-1} a_{m-k},$$

and this is the recurrence relation (2.3.3) for the ξ_m. Thus by taking $\xi_0 = 1$ we have $\xi_m = a_m$, and the series

$$\xi = \sum_{m=0}^{\infty} \varepsilon^m \xi_m$$

converges absolutely for $|\varepsilon| \leq \varepsilon_0$. As a consequence, each series

$$\psi_{jh}(t) = \sum_{m=0}^{\infty} \varepsilon^m \psi_{jh}^{(m)}(t), \qquad j = 1, \ldots, n,$$

converges absolutely and uniformly for all t, $-\infty < t < +\infty$.

We now show that the functions $y_{jh}(t)$ given by (2.2.7) satisfy system (2.2.3). Let us put

$$d_h = \sum_{m=1}^{\infty} \varepsilon^{m-1} S_{mh}. \tag{2.3.6}$$

This series is absolutely convergent for $|\varepsilon|$ sufficiently small since, by (2.3.2), it has the following majorant

$$T^{-1} n M \sum_{m=1}^{\infty} \xi_{m-1} \varepsilon^{m-1},$$

and the latter has been shown to be convergent.

By (2.2.6) we have for $j \neq h$, $j = 1, \ldots, k+1$,

$$\sum_{m=0}^{\infty} \varepsilon^m \left(\frac{d x_{jh}^{(m)}}{dt} \right) = \tau_j \sum_{m=1}^{\infty} \varepsilon^m x_{jh}^{(m)} + \sum_{m=1}^{\infty} \varepsilon^m \sum_{r=1}^{n} \varphi_{jr} x_{rh}^{(m-1)}, \tag{2.3.7}$$

where the series are all absolutely convergent for $|\varepsilon| \leqq \varepsilon_0$; thus the first series (2.3.7) is the derivative of y_j, and hence we have for all t

$$\frac{d y_j}{dt} = \tau_j y_j + \sum_{r=1}^{n} \varphi_{jr} y_r.$$

Analogous computations hold for $j = k+2, \ldots, n$. For $j = h$, we have

$$\sum_{m=0}^{\infty} \varepsilon^m \frac{d x_{hh}^{(m)}}{dt} = \tau_h \sum_{m=0}^{\infty} \varepsilon^m x_{hh}^{(m)} + \sum_{m=1}^{\infty} \varepsilon^m \sum_{r=1}^{n} \varphi_{hr} x_{rh}^{(m-1)} - \sum_{m=1}^{\infty} \varepsilon^m \sum_{l=1}^{m} S_{lh} x_{hh}^{(m-l)}. \tag{2.3.8}$$

The first two series in the right hand member are absolutely convergent. The third series is also absolutely convergent as the product of the two absolutely convergent series

$$\sum_{m=1}^{\infty} \varepsilon^m S_{mh}, \quad \sum_{m=1}^{\infty} \varepsilon^m x_{hh}^{(m)}.$$

Thus the series in the left hand member of (2.3.8) is the derivative of y_h, and we have for all t

$$\frac{d y_h}{dt} = \tau_h y_h - \varepsilon d_h y_h + \sum_{r=1}^{n} \varphi_{hr} y_r.$$

Thereby we have shown that the functions (2.2.7) satisfy (2.2.3). The same functions will satisfy (2.2.1) also if, having chosen $\tau_j = \varrho_j$, $j \neq h$, $j = 1, \ldots, n$, we can determine $\tau_h = \tau_h(\varepsilon)$ in such a way that

$$F(\tau_h, \varepsilon) \equiv \tau_h - \varepsilon d_h(\tau_h, \varepsilon) - \varrho_h = 0. \tag{2.3.9}$$

Now F is an analytic function of τ_h and ε and $F(\varrho_h, 0) = 0$, $F_{\tau_h}(\varrho_h, 0) = 1$. Thus, by the implicit function theorem, there exists for all $|\varepsilon| \leqq \varepsilon_0$ and $\varepsilon_0 > 0$ sufficiently small, a function $\tau_h(\varepsilon)$, $|\varepsilon| \leq \varepsilon_0$, with $\tau_h(0) = \varrho_h$, satisfying (2.3.9) and given by a series

$$\tau_h = \varrho_h + \sum_{r=1}^{\infty} a_{hr} \varepsilon^r$$

which is absolutely convergent for $|\varepsilon| \leqq \varepsilon_0$.

Remark. Each $\tau_h(\varepsilon)$ is an analytic function of ε for all $|\varepsilon|$ sufficiently small, and thus the same holds for the numbers S_{mh} and for the Fourier coefficients of the functions $\psi_{jh}^{(m)}$, $j = 1, \ldots, n$, $m = 0, 1, \ldots$, as can be proved by induction. Also we have $\tau_h = i\alpha_h$, $R(S_{mh}) = 0$ for $\varepsilon = 0$, $h = 1, \ldots, k$, $m = 1, 2, \ldots$.

2.4. Expressions for the solutions and the characteristic exponents. We may now conclude the considerations of the previous sections (2.1), (2.2), (23.). Under the hypotheses (2.1) and for all $|\varepsilon| \leqq \varepsilon_0$ and some $\varepsilon_0 > 0$ sufficiently small, the system (2.1.1) has $n-k$ characteristic exponents

$$\alpha_h(\varepsilon) = \tau_h = \varrho_h + o(1) \quad \text{as} \quad \varepsilon \to 0, \qquad h = k+1, \ldots, n, \tag{2.4.1.a}$$

and k characteristic exponents

$$\alpha_h(\varepsilon) = \tau_h = \varrho_h + \varepsilon d_h(\varepsilon), \qquad h = 1, \ldots, k, \tag{2.4.1.b}$$

where

$$d_h = \sum_{m=1}^{\infty} \varepsilon^{m-1} S_{mh}, \tag{2.4.2}$$

$$S_{mh} = m\left[e^{-\tau_h t} \sum_{r=1}^{n} \varphi_{hr}(t)\, x_{rh}^{(m-1)}\right] = T^{-1} \int_0^T e^{-\tau_h t} \sum_{r=1}^{n} \varphi_{hr}(t)\, x_{rh}^{(m-1)}(t)\, dt, \tag{2.4.3}$$

and

$$\begin{aligned}
&x_{hh}^{(0)} = e^{\tau_h t}, \qquad x_{jh}^{(0)} = 0, \qquad j \neq h, \quad j = 1, \ldots, n, \\
&x_{hh}^{(m)} = e^{\tau_h t} \int e^{-\tau_h t} \left[\sum_{r=1}^{n} \varphi_{hr}\, x_{rh}^{(m-1)} - \sum_{l=1}^{m} S_{lh}\, x_{hh}^{(m-l)}\right] dt, \\
&x_{jh}^{(m)} = e^{\varrho_j t} \int e^{-\varrho_j t} \sum_{r=1}^{n} \varphi_{jr}\, x_{rh}^{(m-1)}\, dt, \qquad j \neq h, \quad j = 1, 2, \ldots, k+1, \\
&x_{jh}^{(m)} = e^{\varrho_j t} \int e^{-\varrho_j t} \left[\sum_{r=1}^{n} \varphi_{jr}\, x_{rh}^{(m-1)} + \delta_{j-1}\, x_{j-1,h}^{(m)}\right] dt, \qquad j = k+2, \ldots, n.
\end{aligned} \tag{2.4.4}$$

The corresponding solutions $y_h = (y_{1h}, \ldots, y_{nh})$ of (2.1.1) are given by

$$y_{jh} = \sum_{m=0}^{\infty} \varepsilon^m x_{jh}^{(m)}, \qquad j = 1, 2, \ldots, n,$$

where h is any number $h = 1, \ldots, k$. All integrals above are the unique primitives of mean value zero considered in (1.2.i), since the integrands also have mean value zero.

If we assume

$$\varphi_{jh}(t) = \sum_{l=0}^{\infty} (a_{jhl} \cos l\omega t + b_{jhl} \sin l\omega t), \qquad j, h = 1, \ldots, n,$$
$$(a_{jhl}, b_{jhl} \text{ real or complex, } b_{jh0} = 0),$$

then

$$S_{1h} = m\left[e^{-\tau_h t} \sum_{j=1}^{n} \varphi_{hj}\, x_{jh}^{(0)}\right] = m(\varphi_{hh}) = a_{hh0}, \qquad h = 1, \ldots, k, \tag{2.4.5}$$

$$\begin{aligned}
x_{hh}^{(1)} = e^{\tau_h t} \psi_{hh}^{(1)} &= e^{\tau_h t} \int [\varphi_{hh}(t) - m(\varphi_{hh})]\, dt \\
&= e^{\tau_h t} \sum_{l=1}^{\infty} (l\omega)^{-1} (-b_{hhl} \cos l\omega t + a_{hhl} \sin l\omega t), \qquad h = 1, \ldots, k,
\end{aligned}$$

and if $\delta_{j-1} = 0$, $j = 2, \ldots, n$, also

$$\begin{aligned}
x_{jh}^{(1)} = e^{\tau_h t} \psi_{jh}^{(1)} = e^{\varrho_j t} \int e^{(\tau_h - \varrho_j) t} \varphi_{jh}(t)\, dt = e^{\tau_h t} \sum_{l=0}^{\infty} [(\tau_h - \varrho_j)^2 + (l\omega)^2]^{-1} \times \\
\times \{[(\tau_h - \varrho_j)\, a_{jhl} - l\omega\, b_{jhl}] \cos l\omega t + [(\tau_h - \varrho_j)\, b_{jhl} + l\omega\, a_{jhl}] \sin l\omega t\}, \\
j = 1, \ldots, n, \quad j \neq h, \quad h = 1, \ldots, k,
\end{aligned}$$

$$\begin{aligned}
S_{2h} &= m\left[\sum_{j=1}^{n} \varphi_{hj}\, \psi_{jh}^{(1)}\right] \\
&= \frac{1}{2} \sum_{\substack{j=1 \\ j \neq h}}^{n} \left\{\frac{2 a_{jh0}\, a_{hj0}}{\tau_h - \varrho_j} + \sum_{l=1}^{\infty} [(\tau_h - \varrho_j)^2 + (l\omega)^2]^{-1} [(\tau_h - \varrho_j)\, C_{hjl} + l\omega\, D_{hjl}]\right\},
\end{aligned} \tag{2.4.6}$$

where $h=1,\ldots,k$, $C_{hjl}=a_{hjl}a_{jhl}+b_{hjl}b_{jhl}$, $D_{hjl}=a_{jhl}b_{hjl}-b_{jhl}a_{hjl}$. For $\varepsilon=0$ the same expression holds with $\tau_h=\varrho_h$. If we take $\varrho_j-\tau_h=\alpha_{hj}+i\beta_{hj}$, α_{hj}, β_{hj} real, then, for $h=1,\ldots,k$, we have

$$
\begin{aligned}
S_{2h}\big|_{\varepsilon=0} = \frac{1}{2}\sum_{\substack{j=1\\ j\neq h}}^{n}\Bigg\{&\frac{-2\alpha_{hj}a_{hj0}a_{jh0}}{\alpha_{hj}^2+\beta_{hj}^2}+\\
&+\sum_{l=1}^{\infty}\left[\frac{-\alpha_{hj}(\alpha_{hj}^2+\beta_{hj}^2+l^2\omega^2)C_{hjl}+l\omega(\alpha_{hj}^2-\beta_{hj}^2+l^2\omega^2)D_{hjl}}{(\alpha_{hj}^2-\beta_{hj}^2+l^2\omega^2)^2+4\alpha_{hj}^2\beta_{hj}^2}\right]\Bigg\}+\\
&+\frac{i}{2}\sum_{\substack{j=1\\ j\neq h}}^{n}\Bigg\{\frac{2\beta_{hj}a_{hj0}a_{jh0}}{\alpha_{hj}^2+\beta_{hj}^2}-\\
&-\sum_{l=1}^{\infty}\frac{\beta_{hj}(-\alpha_{hj}^2-\beta_{hj}^2+l^2\omega^2)C_{hjl}+2l\omega\alpha_{hj}D_{hjl}}{(\alpha_{hj}^2-\beta_{hj}^2+l^2\omega^2)^2+4\alpha_{hj}^2\beta_{hj}^2}\Bigg\}.
\end{aligned}
\tag{2.4.7}
$$

We have also, for the characteristic exponents $\alpha_h(\varepsilon)$, $h=1,2,\ldots,k$,

$$\alpha_h(\varepsilon)=\varrho_h+\varepsilon\, m[\varphi_{hh}]+\varepsilon^2(S_{2h}|_{\varepsilon=0})+O(\varepsilon^3).$$

§ 3. Boundedness theorems

We shall prove first two theorems concerning complex systems of first order differential equations. We shall then apply these results to real systems of first and second order differential equations, since usual transformations reduce these to the systems above.

3.1. Canonical systems of first order linear differential equations

(3.1.i) *Consider the system of first order linear differential equations*

$$\frac{dy}{dt}=Ay+\varepsilon\Phi y, \tag{3.1.1}$$

where $y=\mathrm{col}(y_1,\ldots,y_n)$, $A=\mathrm{diag}(i\sigma_1,\ldots,i\sigma_n)$, $\sigma_1,\ldots,\sigma_n$ *real*, $\sigma_j\neq\sigma_h\ (\mathrm{mod}\ \omega)$, $j\neq h$, $j,h=1,\ldots,n$, *where* Φ *is an* $n\times n$ *matrix whose elements* $\varphi_{jh}(t)$ *are complex-valued functions, periodic of period* $T=2\pi/\omega$, *L-integrable in* $[0,T]$, *with*

$$\varphi_{jh}(t)\sim i\sum_{k=0}^{\infty}a_{jhk}\cos k\omega t,\qquad a_{jhk},\ \textit{real},$$

then for ε *real and sufficiently small in absolute value all solutions of* (3.1.1) *are bounded in* $(-\infty,+\infty)$.

Proof. We shall prove (3.1.i) by showing that the characteristic exponents are purely imaginary. Using the algorithm of § 2 with $k=n$ we have, for each $h=1,\ldots,n$,

$$\tau_h=i\sigma_h+\varepsilon\sum_{k=1}^{\infty}\varepsilon^{k-1}S_{kh}, \tag{3.1.2}$$

where, by (2.2.5), $S_{1h}=m[\varphi_{hh}]=ia_{hh0}$. By the remark at the end of § 2.3 all τ_h, S_{mh}, $h=1,\ldots,n$, $m=1,2,\ldots$, are, for $|\varepsilon|$ sufficiently small, analytic functions of ε with at most branch points of finite order. Also, $R(\tau_h)$, $R(S_{mh})$ are

zero for $\varepsilon = 0$. Thus each of the functions $R(\tau_h)$, $R(S_{mh})$ is either identically zero for all $|\varepsilon|$ sufficiently small, or $O(\varepsilon^\nu)$ for some maximal integer ν (not necessarily the same for all these functions). We shall prove that the first alternative holds for all τ_h, S_{mh}. Indeed, this will be an immediate consequence of the statement above and of the following one, which we shall prove. *For every integer* $\nu = 1, 2, \ldots,$

$$(\mathrm{a}_1) \qquad R(\tau_h), R(S_{mh}) = O(\varepsilon^\nu) \qquad \textit{for all } h = 1, \ldots, n, \quad m = 1, \ldots, \nu,$$

$$(\mathrm{a}_2) \qquad x_{jh}^{(m)} = e^{\tau_h t}\left[\sum_{k=1}^{\infty} (a_{jhk}^{(m)} \cos k\omega t + i\, b_{jhk}^{(m)} \sin k\omega t) + \mu_{jh}^{(m)}(t)\right],$$

where all coefficients $a_{jhk}^{(m)}$, $b_{jhk}^{(m)}$ *are real, and* $\mu_{jh}^{(m)}(t)$ *is a periodic function of period* T, *continuous in* $(-\infty, +\infty)$ *with* $\mu_{jh}^{(m)}(t) = O(\varepsilon^\nu)$ *uniformly in* t, $j, h = 1, \ldots, n$, $k = 1, 2, \ldots,$ $m = 1, \ldots, \nu$.

We shall prove both (a_1), (a_2) by induction.

Obviously $R(S_{1h}) \equiv 0$, $\tau_h = i\sigma_h + \gamma_h = i\sigma_h' + \beta_h$, σ_h', β_h real, $\gamma_h = O(\varepsilon)$; hence $R(\tau_h) = \beta_h = O(\varepsilon)$, $h = 1, \ldots, n$, and thus (a_1) is proved for $\nu = 1$. Also, by (2.2.6) we have

$$(3.1.3) \qquad \begin{aligned} x_{hh}^{(1)} &= e^{\tau_h t} \int [e^{-\tau_h t} \varphi_{hh} e^{\tau_h t} - m(\varphi_{hh})]\, dt \\ &= i\, e^{\tau_h t} \sum_{k=1}^{\infty} (k\omega)^{-1} a_{hhk} \sin k\omega t = i\, e^{\tau_h t} \sum_{k=1}^{\infty} b_{hhk}^{(1)} \sin k\omega t, \end{aligned}$$

where all $b_{hh}^{(1)}$ are real. Analogously we have for $j \neq h$, $j = 1, \ldots, n$,

$$(3.1.4) \qquad \begin{aligned} x_{jh}^{(1)} &= e^{\varrho_j t} \int e^{(\tau_h - \varrho_j) t} \varphi_{jh}(t)\, dt = e^{\varrho_j t} \int e^{(\tau_h - \varrho_j) t} \sum_{k=0}^{\infty} i\, a_{jhk} \cos k\omega t\, dt \\ &= i\, e^{\tau_h t} \sum_{k=0}^{\infty} [(\tau_h - \varrho_j)^2 + (k\omega)^2]^{-1} [(\tau_h - \varrho_j)\, a_{jhk} \cos k\omega t + k\omega\, a_{jhk} \sin k\omega t] \\ &= e^{\tau_h t}\left[\sum_{k=0}^{\infty} (a_{jhk}^{(1)} \cos k\omega t + i\, b_{jhk}^{(1)} \sin k\omega t) + \mu_{jh}^{(1)}(t)\right], \end{aligned}$$

where all real coefficients $a_{jhk}^{(1)}$, $b_{jhk}^{(1)}$ have been obtained by replacing $\tau_h = i\sigma_h' + \beta_h$ by $i\sigma_h'$, and where, therefore,

$$\mu_{jh}^{(1)}(t) = \sum_{k=0}^{\infty} [\mu_{jhk} \cos k\omega t + \nu_{jhk} \sin k\omega t],$$

$$(3.1.5) \qquad \begin{aligned} \mu_{jhk} &= i\{(\tau_h - \varrho_j)[(\tau_h - \varrho_j)^2 + (k\omega)^2]^{-1} - i(\sigma_h' - \sigma_j)[-(\sigma_h' - \sigma_j)^2 + (k\omega)^2]^{-1}\}\, a_{jhk}, \\ \nu_{jhk} &= i\{k\omega[(\tau_h - \varrho_j)^2 + (k\omega)^2]^{-1} - k\omega[-(\sigma_h' - \sigma_j)^2 + (k\omega)^2]^{-1}\}\, a_{jhk}. \end{aligned}$$

We shall now evaluate $\mu_{jh}^{(1)}(t)$. To do this let us observe that, as in § 2, we have supposed $|\varepsilon| \leq \varepsilon_0$ for some $\varepsilon_0 > 0$ sufficiently small in such a way that $\tau_h - \varrho_j \not\equiv 0$ (mod ωi) for all $j \neq h$, $j, h = 1, \ldots, n$. We may now determine $N \geq 1$ so that

$$[(\tau_h - \varrho_j)^2 + (k\omega)^2]^{-1} \leq N(1+k)^{-2}, \qquad |[-(\sigma_h' - \sigma_j)^2 + (k\omega)^2]^{-1}| \leq N(1+k)^{-2},$$
$$|(k\omega)^2 - i(\tau_h - \varrho_j)(\sigma_h' - \sigma_j)| \leq N(1+k)^2, \qquad |\omega(2\tau_h - 2\varrho_j - \beta_h)| \leq N,$$
$$|a_{jhk}| \leq N,$$

for all $j \neq h$, $j, h = 1, \ldots, n$, and all $|\varepsilon| \leqq \varepsilon_0$. By trivial computation we now have

$$|\mu_{jhk}| = |a_{jhk}| \cdot |[(k\omega)^2 - i(\tau_h - \varrho_j)(\sigma'_h - \sigma_j)](\tau_h - i\sigma'_h)| \times$$
$$\times [(\tau_h - \varrho_j)^2 + (k\omega)^2]^{-1} \cdot [-(\sigma'_h - \sigma_j)^2 + (k\omega)^2]^{-1} \leqq N^5 (1+k)^{-2} |\beta_h|,$$

and

$$|\nu_{jhk}| = |a_{jhk}|\,|\beta_h k\omega(2\tau_h - 2\varrho_j - \beta_h)[(\tau_h - \varrho_j)^2 + (k\omega)^2]^{-1}[-(\sigma'_h - \sigma_j)^2 + (k\omega)^2]^{-1}|$$
$$\leqq N^4 k |\beta_h| (1+k)^{-4}.$$

Thus the series (3.1.5) is absolutely convergent, is the Fourier series of $\mu_{jh}^{(1)}(t)$, and

$$|\mu_{jh}^{(1)}(t)| \leqq 2N^5 |\beta_h| \sum_{k=0}^{\infty} (1+k)^{-2} = O(\varepsilon)$$

uniformly in t. Thus both statements (a_1), (a_2) have been proved for $\nu = 1$.

Let us suppose that both (a_1) and (a_2) hold for all integers $1, \ldots, \nu - 1$, and let us prove them for ν.

By (3.1.2) at once we have $R(\tau_h) = O(\varepsilon^\nu)$. This allows a better evalutation for $x_{hh}^{(1)}$, $x_{jh}^{(1)}$. Indeed by repeating word by word the considerations above where $\beta_h = O(\varepsilon)$ is replaced by $\beta_h = O(\varepsilon^\nu)$ we obtain $\mu_{jh}^{(1)}(t) = O(\varepsilon^\nu)$ for all $j, h = 1, \ldots, n$. Thus (a_1), (a_2) are proved for $m = 1$ and ν. We have now, for $m = 2$,

$$\begin{aligned} S_{mh} &= m\left[e^{-\tau_h t} \sum_{j=1}^{n} \varphi_{hj} x_{jh}^{(m-1)}\right] \\ &= m\left\{\sum_{j=1}^{n}\left(\sum_{k=0}^{\infty} i\, a_{hjk} \cos k\omega t\right)\left[\sum_{k=0}^{\infty}(a_{jhk}^{(m-1)} \cos k\omega t + i\, b_{jhk}^{(m-1)} \sin k\omega t) + \mu_{jh}^{(m-1)}(t)\right]\right\} \end{aligned} \tag*{.6)}$$

for $h = 1, \ldots, n$, where $\mu_{jh}^{(m-1)}(t) = O(\varepsilon^\nu)$ for $m = 2$, and the mean value is the usual one for periodic functions of period T. Thus this mean value is T^{-1} times the definite integral in $[0, T]$ of the expression in braces. The sum of products of the trigonometrical series is the sum of a purely imaginary even function and of a real odd function in t. Hence the real part of its mean value, $\bar{S}_{mh}$, is zero. The part of the expression in braces depending on the functions μ is the sum of products of functions μ which are all $O(\varepsilon^\nu)$ and of L-integrable functions. Hence its mean value $S_{mh} - \bar{S}_{mh}$ is $O(\varepsilon^\nu)$. Thus we have $R(S_{mh}) = R[\bar{S}_{mh} + (S_{mh} - \bar{S}_{mh})] = 0 + O(\varepsilon^\nu) = O(\varepsilon^\nu)$ for all $h = 1, \ldots, n$, and $m = 2$. Also, we have for $m = 2$,

$$\begin{aligned} x_{hh}^{(m)} &= e^{\tau_h t} \int \left[e^{-\tau_h t} \sum_{j=1}^{n} \varphi_{hj} x_{jh}^{(m-1)} - \sum_{l=1}^{m} S_{lh} x_{hh}^{(m-l)}\right] dt \\ &= e^{\tau_h t} \int \left\{\sum_{j=1}^{n}\left(\sum_{k=0}^{\infty} i\, a_{hjk} \cos k\omega t\right)\left[\sum_{k=0}^{\infty}(a_{jhk}^{(m-1)} \cos k\omega t + i\, b_{jhk}^{(m-1)} \sin k\omega t) + \mu_{jh}^{(m-1)}(t)\right] - \right. \\ &\quad \left. - \sum_{l=1}^{m} S_{lh}\left[\sum_{k=0}^{\infty}(a_{hhk}^{(m-l)} \cos k\omega t + i\, b_{hhk}^{(m-l)} \sin k\omega t) + \mu_{hh}^{(m-l)}(t)\right]\right\} dt, \end{aligned} \tag*{7)}$$

where, for $m = 2$, all functions μ are $O(\varepsilon^\nu)$ uniformly in t. We may decompose the last integrand into a sum $A(t) + B(t)$, where $A(t)$ is the sum of the products of the trigonometrical series in the first line plus the sum of the products of the purely imaginary numbers $\bar{S}_{mh}$ above times the trigonometrical series of the

second line. Thus $A(t)$ is what the last integrand becomes if $i\tau_h$ is replaced by its imaginary part $i\sigma'_h$. Since in § 2 we have proved that the integrand $A+B$ above has mean value zero for any value of the indeterminate τ_h in a neighborhood of $i\sigma_h$, we conclude that $m[A+B]=0$, $m[A]=0$, and hence $m[B]=0$. Now $A(t)=A_1(t)+A_2(t)$ is the sum of a purely imaginary even function $A_1(t)$ and of a real odd function, hence $m[A_2]=0$, $m[A_1]=m[A]=0$, and $A(t)$ admits of a unique periodic primitive of mean value zero which is the sum of a purely imaginary odd function and of a real even function. Since $m[B]=0$, also $B(t)$ admits of a unique periodic primitive of mean value zero which can be evaluated by means of (1.2.ii). Actually $B(t)$ is the sum of products of periodic functions or of constants which are all $O(\varepsilon^\nu)$ times L-integrable or continuous functions. Thus by (1.2.ii) that primitive of $B(t)$ is $O(\varepsilon^\nu)$ uniformly in t. In conclusion we have

$$x_{hh}^{(m)}=e^{\tau_h t}\sum_{k=1}^{\infty}\left[a_{hhk}^{(m)}\cos k\omega t+i\,b_{hhk}^{(m)}\sin k\omega t\right]+\mu_{hh}^{(m)}(t),$$

with $a_{hhk}^{(m)}$, $b_{hhk}^{(m)}$ real and $\mu_{hh}^{(m)}(t)=O(\varepsilon^\nu)$ uniformly in t for $m=2$. Finally we have

$$\begin{aligned}x_{jh}^{(m)}&=e^{\varrho_j t}\int e^{(\tau_h-\varrho_j)t}\left(\sum_{l=1}^{n}\varphi_{jl}\,x_{lh}^{(m-1)}\right)dt\\ &=e^{\varrho_j t}\int e^{(\tau_h-\varrho_j)t}\sum_{l=1}^{n}\left(\sum_{k=0}^{\infty}i\,a_{jlk}\cos k\omega t\right)\times\\ &\quad\times\left[\sum_{k=0}^{\infty}\left(a_{lhk}^{(m-1)}\cos k\omega t+i\,b_{lhk}^{(m-1)}\sin k\omega t\right)+\mu_{lh}^{(m-1)}(t)\right]dt,\end{aligned}\tag{3.1.8}$$

where, for $m=2$, all functions μ are $O(\varepsilon^\nu)$ uniformly in t. We decompose the integrand into a sum $A(t)+B(t)$, where $A(t)$ is the sum of the product of the trigonometrical series times the exponential function. Then $m[A(t)]=0$ since $\tau_h-\varrho_j\not\equiv 0 \pmod{\omega i}$ for all $j\neq h$. Also, $A(t)=\exp(\tau_h-\varrho_j)t\cdot A_1(t)$, where the periodic function $A_1(t)$ can be thought of as the sum of a purely imaginary even function plus a real odd function, and thus the unique primitive of $A(t)$ of mean value zero has the form

$$\int e^{(\tau_h-\varrho_j)t}\sum_{k=0}^{\infty}\left(i\,\alpha_{jhk}\cos k\omega t+\beta_{jhk}\sin k\omega t\right)dt,$$

where the trigonometric series is the Fourier series of an L-integrable function in $[0,T]$. By (1.2) and formal integration we have a series of the form

$$\begin{aligned}\sum_{k=0}^{\infty}\left[(\tau_h-\varrho_j)^2+(k\omega)^2\right]^{-1}&\left[(\tau_h-\varrho_j)\,i\,\alpha_{jhk}\cos k\omega t+k\omega\, i\,\alpha_{jhk}\sin k\omega t-\right.\\ &\left.-k\omega\,\beta_{jhk}\cos k\omega t+(\tau_h-\varrho_j)\,\beta_{jhk}\sin k\omega t\right].\end{aligned}$$

We can now repeat the same considerations as for (3.1.4) by performing on the terms in the brackets the same manipulations as on the terms in the brackets of the second line of (3.1.4). The conclusion is that, for $m=2$, we have

$$x_{jh}^{(m)}=e^{\tau_h t}\sum_{k=0}^{\infty}\left[a_{jhk}^{(m)}\cos k\omega t+i\,b_{jhk}^{(m)}\sin k\omega t\right]+\mu_{jh}^{(m)}(t),\tag{3.1.9}$$

with $a_{jhk}^{(m)}$, $b_{jhk}^{(m)}$ real, and $\mu_{jh}^{(m)}(t)=O(\varepsilon^\nu)$ for all $j\neq h$, $j,h=1,\ldots,n$, and $m=2$.

The whole process can now be repeated successively for $m=3, 4, \ldots, \nu$. Thus (a_1), (a_2) are proved for every ν. As noticed this implies $R(\tau_h)$, $R(S_{mh})$ identically zero for all $|\varepsilon|$ sufficiently small, and (3.1.i) is thereby proved.

(3.1.ii) *Consider the system of first order linear differential equations*

$$\frac{dy}{dt} = Ay + \varepsilon \Phi y, \tag{3.1.10}$$

where $y = \mathrm{col}(y_1, \ldots, y_n)$, $A = \mathrm{diag}(\varrho_1, \ldots, \varrho_n)$, $R(\varrho_j)=0$, $j=1, \ldots, k$, $R(\varrho_j)<0$, $j=k+1, \ldots, n$, *for some* $0 \leqq k \leqq n$, $\varrho_j \not\equiv \varrho_h \pmod{\omega i}$ *for all* $j \neq h$, $j, h = 1, \ldots, k$, $\varepsilon > 0$ *a parameter, and* Φ *an* $n \times n$ *matrix whose elements are periodic functions of period* $T = 2\pi/\omega$, *L-integrable in* $[0, T]$. *The following conclusions hold:*

(1) *If* $0 \leqq k \leqq n$, *and* $R[m(\varphi_{hh})]<0$ *for all* $h=1, \ldots, k$, *then for* $\varepsilon > 0$ *sufficiently small all solutions of* (3.1.10) *approach zero as* $t \to +\infty$.

(2) *If* $0 < k \leqq n$, *and* $R[m(\varphi_{hh})] > 0$ *for at least one* $h=1, \ldots, k$, *then for* $\varepsilon > 0$ *sufficiently small infinitely many solutions of* (3.1.10) *are unbounded in* $[0, +\infty)$.

Proof. Systems of the type (3.1.10) are discussed in § 2 and the characteristic exponents are given by

$$\alpha_h = \varrho_h + O(\varepsilon), \qquad h = k+1, \ldots, n,$$
$$\alpha_h = \tau_h = \varrho_h + \varepsilon\, m(\varphi_{hh}) + O(\varepsilon^2), \qquad h = 1, \ldots, k.$$

Since $R(\varrho_h)<0$, we have $R(\alpha_h)<0$ for all $h=k+1, \ldots, n$, and ε sufficiently small. For $h=1, \ldots, k$, we have

$$R(\alpha_h) = \varepsilon R[m(\varphi_{hh})] + O(\varepsilon^2).$$

Hence in case (1) we have $R(\alpha_h)<0$ for all $h=1, \ldots, k$; in case (2) we have $R(\alpha_h)>0$ for at least one $h=1, \ldots, k$, and all $\varepsilon>0$ sufficiently small. Thereby (3.1.ii) is proved.

(3.1.iii) *Consider the system of first order linear differential equations*

$$\frac{dy}{dt} = Ay + \varepsilon \Phi y \tag{3.1.11}$$

where $y = \mathrm{col}(y_1, \ldots, y_n)$, $A = \mathrm{diag}(\varrho_1, \ldots, \varrho_n)$, $R(\varrho_j)=0$, $j=1, \ldots, k$, $R(\varrho_j)<0$, $j=k+1, \ldots, n$, *for some* $0 \leqq k < n$, $\varrho_j \not\equiv \varrho_h \pmod{\omega i}$ *for all* $j \neq h$, $j, h=1, \ldots, k$, ε *is a real parameter and* Φ *is an* $n \times n$ *matrix whose elements* $\varphi_{jh}(t)$ *are periodic functions of period* $T = 2\pi/\omega$, *L-integrable in* $[0, T]$. *The following conclusions hold:*

(1) *If either* (a) $\varphi_{jh} = \mu_j \psi_{jh}$, $\psi_{jh} = -\psi_{hj}$, $m[\varphi_{hh}]=0$ *for all* $j \neq h$, $j, h=1, \ldots, n$, *or* (b) $\varphi_{jh} = i\mu_j \psi_{jh}$, $\psi_{jh} = \psi_{hj}$, $m[\varphi_{hh}]=0$ *for all* $j \neq h$, $j, h = 1, \ldots, n$, *where all functions* ψ *are real, all numbers* μ_j *are real,* $\neq 0$, *and of the same sign, if* $0 \leqq k < n$, *and for every* $h=1, \ldots, k$, *at least one of the functions* φ_{hj}, $j=k+1, \ldots, n$, *is not identically zero, then for* $|\varepsilon| \neq 0$ *sufficiently small all solutions of* (3.1.11) *approach zero as* $t \to +\infty$.

(2) *If either* (a) $\varphi_{jh} = \mu_j \psi_{jh}$, $m[\varphi_{hh}]=0$, $\psi_{jh} = \psi_{hj}$, $m[\varphi_{hh}]=0$ *for all* $j \neq h$, $j, h=1, \ldots, n$, *or* (b) $\varphi_{jh} = i\mu_j \psi_{jh}$, $\psi_{jh} = -\psi_{hj}$ *for all* $j \neq h$, $j, h=1, \ldots, n$, *where all functions* ψ *are real, all numbers* μ_j *are real,* $\neq 0$ *and of the same sign, if* $0<k<n$, *and for at least one* $h=1, \ldots, k$, *at least one of the functions* φ_{hj}, $j=k+1, \ldots, n$, *is not identically zero, then for* $|\varepsilon| \neq 0$ *sufficiently small infinitely many solutions of* (3.1.11) *are unbounded in* $[0, +\infty)$.

Proof. Systems of the type (3.1.11) are discussed in § 2 and the characteristic exponents are given by

$$\alpha_h = \varrho_h + o(1) \quad \text{as} \quad \varepsilon \to 0, \qquad h = k+1, \ldots, n;$$
$$\alpha_h = \varrho_h + \varepsilon\, m(\varphi_{hh}) + \varepsilon^2 S_{2h}|_{\varepsilon=0} + O(\varepsilon^3), \qquad h = 1, \ldots, k.$$

Since $R(\varrho_h) < 0$, we have $R(\alpha_h) < 0$ for all $h = k+1, \ldots, n$ and $|\varepsilon|$ sufficiently small. For $h = 1, \ldots, k$, we have, in all cases (1a), (1b), (2a), (2b):

$$R(\alpha_h) = \varepsilon^2 R(S_{2h}|_{\varepsilon=0}) + O(\varepsilon^3),$$

where $R(S_{2h}|_{\varepsilon=0})$ can be obtained by (2.4.6). By hypothesis we have

$$\varphi_{jh} \sim \sum_{l=0}^{\infty} (a_{jhl} \cos l\omega t + b_{jhl} \sin l\omega t),$$

where a_{jhl}, b_{jhl} are real, $(b_{jh0} = 0)$. In case (1a), from (2.4.6) we deduce

$$(3.1.12) \quad R(S_{2h}|_{\varepsilon=0}) = \frac{1}{2} \sum_{\substack{j=1 \\ j \neq h}}^{n} \left[\frac{2\alpha_{hj}\mu_h\mu_j\alpha_{hj0}^2}{\alpha_{hj}^2 + \beta_{hj}^2} + \sum_{l=1}^{\infty} \frac{\alpha_{hj}(\alpha_{hj}^2 - \beta_{hj}^2 + l^2\omega^2)\mu_j\mu_h(a_{hjl}^2 + b_{hjl}^2)}{(\alpha_{hj}^2 - \beta_{hj}^2 + l^2\omega^2)^2 + 4\alpha_{hj}^2\beta_{hj}^2} \right]$$

where $\varrho_j - \varrho_h = \alpha_{hj} + i\beta_{hj}$, α_{hj}, β_{hj} real, $h = 1, \ldots, k$. Thus $\alpha_{hj} = 0$ for $j = 1, \ldots, k$, $\alpha_{hj} < 0$ for $j = k+1, \ldots, n$, and $R(S_{2h}|_{\varepsilon=0}) \leqq 0$ for each $h = 1, \ldots, k$. Since, for every $h = 1, \ldots, k$, at least one of the functions $\varphi_{hj}(t)$, $j = k+1, \ldots, n$, is not identically zero, we conclude that $R(S_{2h}|_{\varepsilon=0}) < 0$ for every $h = 1, \ldots, k$. Thereby (1a) is proved. The same formula above and the same argument hold in case (1b).

In cases (2a), (2b) the formula above holds where $\frac{1}{2}$ is replaced by $-\frac{1}{2}$. Hence $R(S_{2h}|_{\varepsilon=0}) > 0$ for at least one $h = 1, \ldots, k$.

3.2. Systems of first and second order linear differential equations

(3.2.i) *Consider the system*

$$(3.2.1) \qquad y'' + Ay + \varepsilon\Phi(t)\, y = 0,$$

where $y = \operatorname{col}(y_1, \ldots, y_n)$, $A = \operatorname{diag}(\sigma_1^2, \ldots, \sigma_n^2)$, $\Phi = (f_{jh}(t))$, $j, h = 1, \ldots, n$, *where all* $f_{jh}(t)$, ε *are real and* $\sigma_j > 0$, *where the functions* $f_{jh}(t)$ *are periodic of period* $T = 2\pi/\omega$, *L-integrable in* $[0, T]$, *and all* $2\sigma_j$, $\sigma_j + \sigma_h$, $\sigma_j - \sigma_h \not\equiv 0 \pmod{\omega}$, $j \neq h$, $j, h = 1, \ldots, n$. *If all functions* $f_{jh}(t)$ *are even, then for all* $|\varepsilon|$ *sufficiently small all solutions of* (3.2.1) *are bounded in* $(-\infty, +\infty)$.

Proof. As usual the boundedness of a solution y in (3.2.i) refers to the boundedness of $|y_1| + |y_1'| + \cdots + |y_n'|$. Statement (3.2.i) is a corollary of (3.1.i). Indeed by the transformations $y_j = w_{2j-1}$, $y_j' = w_{2j}$, $j = 1, \ldots, n$, $w_{2j-1} = z_{2j-1} + z_{2j}$, $w_{2j} = i\sigma_j z_{2j-1} - i\sigma_j z_{2j}$, $j = 1, \ldots, n$, system (3.2.i) becomes

$$z_{2j-1}' = i\sigma_j z_{2j-1} - \varepsilon(2i\sigma_j)^{-1} \sum_{h=1}^{n} f_{jh}(t)\,(z_{2h-1} + z_{2h}),$$
$$z_{2j}' = -i\sigma_j z_{2j} + \varepsilon(2i\sigma_j)^{-1} \sum_{h=1}^{n} f_{jh}(t)\,(z_{2h-1} + z_{2h}), \qquad j = 1, \ldots, n,$$

and (3.2.i) follows from (3.1.i).

Remark. Statement (3.2.i) is a particular case of the following, proved by the same method and a different argument by L. CESARI, J. K. HALE, R. A. GAMBILL [*1, 7, 13*]:

(*) Under the same general hypotheses as in (3.2.i), if $\Phi = \Phi_1 + \Phi_2$, where $\Phi_1 = \operatorname{diag}(\Psi_1, \ldots, \Psi_k)$, the non-zero elements of Φ_2 are all below, or all above the matrices $\Psi_1, \ldots, \Psi_k$ in Φ_1, and for each $\Psi_s = (\psi_{uv}^{(s)})$ either $\psi_{uv}^{(s)}(-t) = \psi_{uv}^{(s)}(t)$ for all u, v, or $\psi_{vu}^{(s)}(t) = \psi_{uv}^{(s)}(t)$ for all u, v, then for all $|\varepsilon|$ sufficiently small all solutions of (3.2.i) are bounded.

For other statements including (3.2.i) [but not (*)] see R. A. GAMBILL [*7*] and J. K. HALE [*13, 15*].

(3.2.ii) *Consider the system*

$$y_1' + \varepsilon \sum_{h=1}^{n} f_{1h}(t)\, y_h = 0,$$
$$(3.2.2)\qquad y_j'' + c_j y_j' + d_j y_j + \varepsilon \sum_{h=1}^{n} f_{jh}(t)\, y_h = 0, \qquad j = 2, \ldots, n,$$

where $c_j > 0$, $d_j > 0$, $j = 2, \ldots, n$, ε real, $n \geqq 1$, $f_{jh}(t)$ is a real periodic function of period $T = 2\pi/\omega$, L-integrable in $[0, T]$ with Fourier series

$$f_{jh}(t) \sim \sum_{l=0}^{\infty} (a_{jhl} \cos l\omega t + b_{jhl} \sin l\omega t), \qquad j, h = 1, \ldots, n.$$

If either $a_{110} \neq 0$, $\varepsilon a_{110} > 0$, or $a_{110} = 0$, $M < 0$, $\varepsilon \neq 0$, then for $|\varepsilon|$ sufficiently small, all solutions of (3.2.2) *approach zero as $t \to +\infty$.*

If either $a_{110} \neq 0$, $\varepsilon a_{110} < 0$, or $a_{110} = 0$, $M > 0$, $\varepsilon \neq 0$, then for $|\varepsilon|$ sufficiently small, infinitely many solutions of (3.2.2) *are unbounded in* $[0, +\infty)$.

In both cases M is given by

$$M = \sum_{j=2}^{n} \sum_{l=0}^{\infty} p_{jl} C_{1jl} + q_{jl} D_{1jl},$$
$$p_{j0} = d_j^{-1}, \quad q_{j0} = 0, \quad m_{jl} = [d_j^2 + l^2\omega^2(c_j^2 - 2d_j) + l^4\omega^4]^{-1},$$
$$(3.2.3)\qquad p_{jl} = 2^{-1} m_{jl}(d_j - l^2\omega^2), \qquad q_{jl} = 2^{-1} m_{jl}\, l\omega c_j,$$
$$C_{1jl} = a_{1jl} a_{j1l} + b_{1jl} b_{j1l}, \qquad D_{1jl} = a_{j1l} b_{1jl} - b_{j1l} a_{1jl},$$
$$b_{jh0} = 0, \qquad j, h = 1, \ldots, n, \quad l = 0, 1, \ldots.$$

Proof. As usual, boundedness, unboundedness, and approach to zero refer to $|y_1| + |y_2| + |y_2'| + \cdots + |y_n| + |y_n'|$. Suppose first $c_j^2 \neq 4d$, $j = 2, \ldots, n$. Denote by r_{2j-2}, r_{2j-1} the roots of the equation $r^2 + c_j r + d_j = 0$, $j = 2, \ldots, n$. By the transformations $y_1 = w_1$, $y_j = w_{2j-2}$, $y_j' = w_{2j-1}$, $j = 2, \ldots, n$, and $w_1 = z_1$, $w_{2j-2} = z_{2j-2} + z_{2j-1}$, $w_{2j-1} = r_{2j-2} z_{2j-2} + r_{2j-1} z_{2j-1}$, $j = 2, \ldots, n$, system (3.2.2) becomes the following system of $2n-1$ first order equations:

$$z_1' = -\varepsilon f_{11} z_1 - \varepsilon \sum_{h=2}^{n} f_{1h}(z_{2h-2} + z_{2h-1}),$$
$$(3.2.4)\qquad z_{2j-2}' = r_{2j-2} z_{2j-2} + \varepsilon (r_{2j-1} - r_{2j-2})^{-1} \left[f_{j1} z_1 + \sum_{h=2}^{n} f_{jh}(z_{2h-2} + z_{2h-1}) \right],$$
$$z_{2j-1}' = r_{2j-1} z_{2j-1} - \varepsilon (r_{2j-1} - r_{2j-2})^{-1} \left[f_{j1} z_1 + \sum_{h=2}^{n} f_{jh}(z_{2h-2} + z_{2h-1}) \right],$$
$$j = 2, \ldots, n.$$

If we denote this system by

$$z' = A z + \varepsilon \Phi(t) z,$$

where $z = \mathrm{col}(z_1, \ldots, z_{2n-1})$, $A = \mathrm{diag}(0, r_2, r_3, \ldots, r_{2n-2}, r_{2n-1})$, $\Phi(t) = (\tilde{\varphi}_{jh})$, $j, h = 1, \ldots, 2n-1$, with

$$\tilde{\varphi}_{jh} \sim \sum_{l=0}^{\infty} (\tilde{a}_{jhl} \cos l\omega t + \tilde{b}_{jhl} \sin l\omega t), \qquad j, h = 1, \ldots, 2n-1,$$

then we have, for instance, for all $s = 2, \ldots, n$, $l = 0, 1, \ldots$,

$$\begin{aligned}
&\tilde{a}_{11l} = -a_{11l}, && \tilde{b}_{11l} = -b_{11l},\\
&\tilde{a}_{1,2s-2,l} = -a_{1sl}, && \tilde{b}_{1,2s-2,l} = -b_{1sl},\\
(3.2.5) \qquad &\tilde{a}_{1,2s-1,l} = -a_{1sl}, && \tilde{b}_{1,2s-1,l} = -b_{1sl},\\
&\tilde{a}_{2s-2,1,l} = -\tilde{a}_{2s-1,1,l} = (r_{2s-1} - r_{2s-2})^{-1} a_{s1l}, &&\\
&\tilde{b}_{2s-2,1,l} = -\tilde{b}_{2s-1,1,l} = (r_{2s-1} - r_{2s-2})^{-1} b_{s1l}. &&
\end{aligned}$$

If $\alpha_1(\varepsilon), \ldots, \alpha_{2n-1}(\varepsilon)$ are the $2n-1$ characteristic exponents of system (3.2.4), then we have

$$\alpha_1(\varepsilon) = -\varepsilon m(f_{11}) + O(\varepsilon^2)$$

$$\alpha_j(\varepsilon) = r_j + o(1) \quad \text{as} \quad \varepsilon \to 0, \qquad j = 2, \ldots, 2n-1.$$

Thus $R[\alpha_j(\varepsilon)] < 0$, $j = 2, \ldots, 2n-1$ and $\alpha_1(\varepsilon)$ real for all $|\varepsilon|$ sufficiently small, and $\alpha_1(\varepsilon) < 0$, or > 0 according as $m(f_{11}) \neq 0$, $\varepsilon m(f_{11}) > 0$ or < 0. If we suppose $m(f_{11}) = 0$, then

$$\alpha_1(\varepsilon) = \varepsilon^2 (S_{21}|_{\varepsilon=0}) + O(\varepsilon^3).$$

We have now, for $h = 1$, $\varrho_j = \varrho_1 = r_j$, $j = 2, \ldots, 2n-1$, and, by (2.4.6)

$$M = S_{21}|_{\varepsilon=0} = 2^{-1} \sum_{j=2}^{2n-1} \left[-2 r_j^{-1} \tilde{a}_{1j0} \tilde{a}_{j10} + \sum_{l=1}^{\infty} (r_j^2 + l^2 \omega^2)^{-1} (-r_j \tilde{C}_{1jl} + l\omega \tilde{D}_{1jl}) \right].$$

By (3.2.5) we have

$$\tilde{C}_{1,2s-2,l} = -\tilde{C}_{1,2s-1,l} = -(r_{2s-1} - r_{2s-2})^{-1} C_{1sl},$$

$$\tilde{D}_{1,2s-2,l} = -\tilde{D}_{1,2s-2,l} = -(r_{2s-1} - r_{2s-2})^{-1} D_{1sl}, \qquad s = 2, \ldots, n,$$

and, then, by trivial computations, and by remarking that $r_{2s-2} r_{2s-1} = d_s$, $r_{2s-2} + r_{2s-1} = -c_s$, we obtain (3.2.3). If, for some $j = 2, \ldots, n$ we have $c_j^2 = 4 d_j$, formula (3.2.3) still holds, since we can regard $S_{2h}|_{\varepsilon=0}$ as a continuous function of c_j, d_j, and thus we can obtain the same formula by a passage to the limit.

Examples. The equation $y' + \varepsilon f(t) y = 0$ has solutions $y = C \exp\left(-\varepsilon \int_0^t f(s)\, ds\right)$, and thus they all approach zero as $t \to +\infty$ if $m[f] \neq 0$, $\varepsilon m[f] > 0$, or all but the trivial solution are unbounded in $[0, +\infty)$ if $m[f] \neq 0$, $\varepsilon m[f] < 0$.

Consider now the system

$$y_1' + (\varepsilon \cos t \cdot y_1 + \varepsilon \sin t \cdot y_2) = 0,$$

$$y_2'' + 3 y_2' + 2 y_2 + (-\varepsilon \sin t \cdot y_1 + \varepsilon \cos t \cdot y_2) = 0,$$

Here $f_{11} = \cos t$, $f_{12} = \sin t$, $f_{21} = -\sin t$, $f_{22} = \cos t$, $d_2 = 2$, $c_2 = 3$, and hence $C_{121} = -1$, $D_{121} = 0$, and $M = -20^{-1} < 0$. All solutions of this system approach zero as $t \to +\infty$.

Consider now the system

$$y_1' + (\varepsilon \cos t \cdot y_1 + \varepsilon \sin t \cdot y_2) = 0,$$
$$y_2'' + 3y_2' + 2y_2 + (\varepsilon \cos t \cdot y_1 + \varepsilon \cos t \cdot y_2) = 0$$

with $f_{11} = \cos t$, $f_{12} = \sin t$, $f_{21} = \cos t$, $f_{22} = \cos t$, and hence $C_{121} = 0$, $D_{121} = 1$, and $M = 3 \cdot 20^{-1} > 0$. Thus infinitely many solutions of the last system are unbounded in $[0, +\infty)$.

(3.2.iii) *Consider the system*

$$\text{(3.2.6)} \qquad \begin{aligned} &y_1'' + a y_1' + \varepsilon \sum_{h=1}^{n} f_{1h}(t)\, y_h = 0, \\ &y_j'' + c_j y_j' + d_j y_j + \varepsilon \sum_{h=1}^{n} f_{jh}(t)\, y_h = 0, \qquad j = 2, \ldots, n, \end{aligned}$$

where $a > 0$, $c_j > 0$, $d_j > 0$, $j = 2, \ldots, n$, ε real, $n \geqq 1$, $f_{jh}(t)$ real periodic functions of period $T = 2\pi/\omega$, L-integrable in $[0, T]$ with Fourier series

$$f_{jh}(t) \sim \sum_{l=0}^{\infty} (a_{jhl} \cos l\omega t + b_{jhl} \sin l\omega t), \qquad j, h = 1, \ldots, n.$$

If either $a_{110} \neq 0$, $\varepsilon a_{110} > 0$ or $a_{110} = 0$, $M < 0$, $\varepsilon \neq 0$, then for $|\varepsilon|$ sufficiently small, all solutions of (3.2.6) approach zero as $t \to +\infty$.

If either $a_{110} \neq 0$, $\varepsilon a_{110} < 0$ or $a_{110} = 0$, $M > 0$, $\varepsilon \neq 0$, then for $|\varepsilon|$ sufficiently small, infinitely many solutions of (3.2.6) are unbounded in $[0, +\infty)$. In both cases M is given by

$$M = \sum_{j=1}^{n} \sum_{l=0}^{\infty} (p_{jl} C_{1jl} + q_{jl} D_{1jl}),$$

$$\text{(3.2.7)} \qquad \begin{aligned} &p_{10} = -a^{-3}, \qquad p_{1l} = -(2a)^{-1}(a^2 + l^2\omega^2)^{-1}, \\ &q_{10} = q_{1l} = 0, \qquad p_{j0} = a^{-1} d_j^{-1}, \qquad q_{j0} = 0, \\ &m_{jl} = [d_j^2 + l^2\omega^2(c_j^2 - 2d_j) + l^4\omega^4]^{-1}, \qquad j = 2, \ldots, n, \\ &p_{jl} = -(2a)^{-1} m_{jl}(l^2\omega^2 - d_j), \quad q_{jl} = (2a)^{-1} m_{jl} c_j l\omega, \\ &\qquad\qquad j = 2, \ldots, n, \quad l = 1, 2, \ldots, \\ &b_{jh0} = 0, \quad C_{1jl} = a_{1jl} a_{j1l} + b_{1jl} b_{j1l}, \quad D_{1jl} = a_{j1l} b_{1jl} - b_{j1l} a_{1jl}, \\ &\qquad\qquad j, h = 1, \ldots, n, \quad l = 1, \ldots. \end{aligned}$$

If $n = 1$, and $a_{110} = 0$, then we have necessarily $M < 0$, and the first alternative holds.

Proof. As usual boundedness, unboundedness, approach to zero refer to $|y_1| + |y_1'| + \cdots + |y_n| + |y_n'|$. Suppose $c_j^2 \neq 4d_j$, $j = 2, \ldots, n$. Denote by r_{2j-1}, r_{2j} the roots of the equation $r^2 + c_j r + d_j = 0$, $j = 2, \ldots, n$. By the transformations $y_j = w_{2j-1}$, $y_j' = w_{2j}$, $j = 1, \ldots, n$, and $w_1 = z_1 + z_2$, $w_2 = a z_2$, $w_{2j-1} = z_{2j-1} + z_{2j}$,

$w_{2j} = r_{2j-1} z_{2j-1} + r_{2j} z_{2j}$, system (3.2.6) becomes

$$
\begin{aligned}
z_1' &= -\varepsilon a^{-1} \sum_{h=1}^{n} f_{1h}(z_{2h-1} + z_{2h}), \\
z_2' &= -a z_2 + \varepsilon a^{-1} \sum_{h=1}^{n} f_{1h}(z_{2h-1} + z_{2h}), \\
z_{2j-1}' &= r_{2j-1} z_{2j-1} - \varepsilon (r_{2j} - r_{2j-1})^{-1} \sum_{h=1}^{n} f_{jh}(z_{2h-1} + z_{2h}), \\
z_{2j}' &= r_{2j} z_{2j} + \varepsilon (r_{2j} - r_{2j-1})^{-1} \sum_{h=1}^{n} f_{jh}(z_{2h-1} + z_{2h}), \qquad j = 2, \ldots, n,
\end{aligned}
\tag{3.2.8}
$$

or $z' = Az + \varepsilon \Phi z$, where $z = (z_1, \ldots, z_{2n})$, $A = \operatorname{diag}(0, -a, r_2, \ldots, r_{2n})$, $\Phi(t) = (\tilde{\varphi}_{jh}(t))$, $j, h = 1, \ldots, 2n$, with $\tilde{\varphi}_{jh}(t) \sim \sum_{l=0}^{\infty} (\tilde{a}_{jhl} \cos l\omega t + \tilde{b}_{jhl} \sin l\omega t)$, and, for instance,

$$
\begin{aligned}
\tilde{a}_{1,2s-1,l} = \tilde{a}_{1,2s,l} &= -a^{-1} a_{1sl}, && s = 1, 2, \ldots n, \\
\tilde{a}_{2,2s-1,l} = \tilde{a}_{2,2s,l} &= a^{-1} a_{1sl}, && s = 1, 2, \ldots, n, \\
\tilde{a}_{2s-1,1,l} &= -(r_{2s} - r_{2s-1})^{-1} a_{s1l}, && s = 2, \ldots, n, \\
\tilde{a}_{2s,1,l} &= (r_{2s} - r_{2s-1})^{-1} a_{s1l}, && s = 2, \ldots, n.
\end{aligned}
$$

Analogous relations hold for the coefficients $\tilde{b}, b$. If $\alpha_1(\varepsilon), \ldots, \alpha_{2n}(\varepsilon)$ are the $2n$ characteristic exponents of system (3.2.8) we have

$$
\begin{aligned}
&\alpha_1(\varepsilon) = -\varepsilon a^{-1} m(f_{11}) + O(\varepsilon^2), \quad \alpha_2(\varepsilon) = -a + o(1) \quad \text{as} \quad \varepsilon \to 0, \\
&\alpha_j(\varepsilon) = r_j + o(1) \quad \text{as} \quad \varepsilon \to 0, \qquad j = 3, \ldots, 2n.
\end{aligned}
$$

Thus, for all $|\varepsilon|$ sufficiently small, we have $R[\alpha_j(\varepsilon)] < 0$, $j = 2, \ldots, n$, and $\alpha_1(\varepsilon)$ is real and corresponds to a simple multiplier. If $m(f_{11}) \neq 0$, then $R[\alpha_1(\varepsilon)] < 0$ $[>0]$ for all $|\varepsilon|$ sufficiently small with $\varepsilon m[f_{11}] > 0$ $[<0]$. If $m[f_{11}] = 0$ then

$$\alpha_1(\varepsilon) = \varepsilon^2 (S_{21}|_{\varepsilon=0}) + O(\varepsilon^3).$$

We have now, for $h = 1$, $\varrho_2 - \varrho_1 = -a - 0 = -a$, $\alpha_{12} = -a$, $\beta_{12} = 0$, $\varrho_j - \varrho_1 = r_j$, $j = 3, \ldots, 2n$, and, by (2.4.6),

$$
\begin{aligned}
M = R(S_{21}|_{\varepsilon=0}) = \tfrac{1}{2} R \Big\{ 2a^{-1} \tilde{a}_{120} \tilde{a}_{210} + \sum_{l=1}^{\infty} (a^2 + l^2\omega^2)^{-1} (a \tilde{C}_{12l} + l\omega \tilde{D}_{12l}) \Big\} + \\
+ \tfrac{1}{2} R \sum_{j=3}^{2n} \Big\{ 2 r_j^{-1} \tilde{a}_{1j0} \tilde{a}_{j10} + \sum_{l=1}^{\infty} (r_j^2 + l^2\omega^2)^{-1} (-r_j \tilde{C}_{1jl} + l\omega \tilde{D}_{1jl}) \Big\}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\tilde{C}_{12l} &= -a^{-2}(a_{11l}^2 + b_{11l}^2), \quad \tilde{D}_{12l} = 0, \\
\tilde{C}_{1,2s-1,l} &= -\tilde{C}_{1,2s,l} = a^{-1}(r_{2s} - r_{2s-1})^{-1} C_{1sl}, \\
\tilde{D}_{1,2s-1,l} &= -\tilde{D}_{1,2s,l} = a^{-1}(r_{2s} - r_{2s-1})^{-1} D_{1sl}, \qquad s = 2, \ldots, n
\end{aligned}
$$

by trivial computations we have (3.2.7).

If $n = 1$, then M is given in the first line of (3.2.7) and thus $M < 0$, and $\alpha_1(\varepsilon) < 0$ for all $|\varepsilon| \neq 0$ sufficiently small.

Remark. For $n=1$, system (3.2.8) becomes

$$z_1' = \varepsilon a^{-1}(-f_{11}z_1 - f_{11}z_2), \qquad z_2' = -a z_2 + \varepsilon a^{-1}(f_{11}z_1 + f_{11}z_2),$$

and the contention of (3.2.iii) is a consequence of (3.1.iii), case (1a), with $\mu_1=\mu_2=1$.

Examples. The equation $y''+ay'+\varepsilon f(t)y=0$, $a>0$, has all solutions $y(t)$ approaching zero as $t\to+\infty$ [with $y'(t)$]. Consider the system

$$y_1''+a y_1'+\varepsilon f_{11}(t)\, y_1+\varepsilon f_{12}(t)\, y_2=0,$$

$$y_2''+c_2 y_2'+d_2 y_2+\varepsilon f_{21}(t)\, y_1+\varepsilon f_{22}(t)\, y_2=0,$$

with $m[f_{11}]=C_{110}\neq 0$. Then for $|\varepsilon|$ sufficiently small, $\varepsilon C_{110}>0$, all its solutions approach zero as $t\to+\infty$.

Consider now the system

$$y_1''+y_1'+\varepsilon\cos t\cdot y_1+\varepsilon\cos t_2\, y_2=0,$$

$$y_2''+y_2'+y_2+\varepsilon\cos t\cdot y_1+\varepsilon\cos t\cdot y_2=0.$$

We have here $m[\cos t]=0$, $M=-\frac{1}{4}<0$, and all its solutions approach zero as $t\to+\infty$ provided $|\varepsilon|$ is sufficiently small.

Consider the system

$$y_1''+y_1'+\varepsilon\cos t\cdot y_1+\varepsilon\sin t\cdot y_2=0,$$

$$y_2''+y_2'+y_2+\varepsilon\cos t\cdot y_1+\varepsilon\cos t\cdot y_2=0.$$

We have here $m[\cos t]=0$, $M=\frac{1}{4}>0$ and infinitely many of its solutions are unbounded in $[0,+\infty)$. The same holds for the system

$$y_1''+y_1'+\varepsilon\cos t\cdot y_1+\varepsilon\cos t\cdot y_2=0,$$

$$y_2''+y_2'+y_2-\varepsilon\sin t\cdot y_1+\varepsilon\cos t\cdot y_2=0,$$

with $m[\cos t]=0$, $M=\frac{1}{4}>0$.

Both (3.2.ii) and (3.2.iii) are particular cases of the statements (3.2.v), (3.2.vi). We give below a few statements which show how the previous ones can be extended. They are all corollaries of (2.4).

(3.2.iv) *Consider the system*

$$
\begin{aligned}
&y_j''+\sigma_j^2 y_j+\varepsilon\sum_{h=1}^{n} f_{jh}(t)\, y_h=0, && \sigma_j>0,\quad j=1,\dots,k,\\
&y_j''+c_j y_j'+d_j y_j+\varepsilon\sum_{h=1}^{n} f_{jh}(t)\, y_h=0, && c_j,\quad d_j>0,\quad j=k+1,\dots,n
\end{aligned}
\tag{3.2.9}
$$

where $1\leq k<n$, $2\sigma_j$, $\sigma_j+\sigma_h$, $\sigma_j-\sigma_h\not\equiv 0 \pmod{\omega}$, $j\neq h$, $j,h=1,\dots,k$, ε *real,* $f_{jh}(t)$ *real periodic function of period* $T=2\pi/\omega$, *L-integrable in* $[0,T]$, *with Fourier series*

$$f_{jh}(t)\sim\sum_{l=0}^{\infty}(a_{jhl}\cos l\omega t+b_{jhl}\sin l\omega t), \qquad j,h=1,\dots,n.$$

If $M_h<0$, *for all* $h=1,\dots,k$, *then for all* $|\varepsilon|\neq 0$ *sufficiently small all solutions of* (3.2.9) *approach zero as* $t\to+\infty$. *If* $M_h>0$ *for at least one* $h=1,\dots,k$, *then for all* $|\varepsilon|\neq 0$ *sufficiently small infinitely many solutions of* (3.2.9) *are unbounded in*

$[0, +\infty)$. *In both cases* M_h *is given by*

$$M_h = \sum_{\substack{j=1 \\ j \neq h}}^{n} \sum_{l=0}^{\infty} (p_{hjl} C_{hjl} + q_{hjl} D_{hjl}), \qquad h = 1, \ldots, k,$$

where

$$p_{hj0} = q_{hj0} = 0, \quad p_{hjl} = 0, \quad q_{hjl} = -2^{-1} l\omega [4\sigma_h^2 \sigma_j^2 - (l^2\omega^2 - \sigma_j^2 - \sigma_h^2)^2]^{-1},$$
$$j = 1, \ldots, k, \quad j \neq h, \quad l = 1, 2, \ldots;$$

$$p_{hj0} = -2^{-1} c_j [(d_j - \sigma_h^2)^2 + \sigma_h^2 c_j^2]^{-1}, \qquad q_{hj0} = 0, \qquad j = k+1, \ldots, n;$$

$$p_{hjl} = -4^{-1} c_j (\alpha_{hjl}^2 - 4\gamma_{hjl}) M_{hjl}^{-1}, \qquad q_{hjl} = 2^{-1} l\omega (\beta_{hjl}^2 - 4\delta_{hjl}) M_{hjl}^{-1},$$

$$M_{hjl} = [\beta_{hjl}^2 + (c_j^2 - 4d_j)\sigma_h^2 + c_j^2(4^{-1} c_j^2 - d_j + \sigma_h^2)]^2 - (c_j^2 - 4d_j)\sigma_h^2 (2\beta_{hjl} - c_j^2)^2,$$

$$\alpha_{hjl} = \sigma_h^2 + 2^{-1} c_j^2 - d_j + l^2\omega^2, \qquad \beta_{hjl} = -\sigma_h^2 + d_j + l^2\omega^2,$$

$$\gamma_{hjl} = l^4\omega^4 + 4^{-1} c_j^2 l^2 \omega^2 + \sigma_h^2 (4^{-1} c_j^2 - d_j),$$

$$\delta_{hjl} = (16)^{-1} c_j^4 + 4^{-1} c_j^2 l^2 \omega^2 + \sigma_h^2 (4^{-1} c_j^2 - d_j), \qquad j = k+1, \ldots, n, \quad l = 1, 2, \ldots;$$

$$C_{hjl} = a_{hjl} a_{jhl} + b_{hjl} b_{jhl}, \qquad D_{hjl} = a_{jhl} b_{hjl} - b_{jhl} a_{hjl},$$
$$h, j = 1, \ldots, n, \quad l = 0, 1, \ldots,$$

$$b_{hj0} = 0, \qquad h, j = 1, \ldots, n.$$

(3.2.v) *Consider the system*

$$\begin{aligned} & y_1' + \varepsilon \sum_{h=1}^{n} f_{1h}(t)\, y_h = 0, \\ (3.2.10) \qquad & y_j'' + \sigma_j^2 y_j + \varepsilon \sum_{h=1}^{n} f_{jh}(t)\, y_h = 0, \qquad \sigma_j > 0, \quad j = 2, \ldots, k, \\ & y_j'' + c_j y_j' + d_j y_j + \varepsilon \sum_{h=1}^{n} f_{jh}(t)\, y_h = 0, \qquad c_j, \quad d_j > 0, \quad j = k+1, \ldots, n, \end{aligned}$$

where $2 \leq k < n$, $2\sigma_j$, $\sigma_j + \sigma_h$, $\sigma_j - \sigma_h \not\equiv 0 \pmod{\omega}$, $j \neq h$, $j, h = 2, \ldots, k$, ε *real*, $f_{jh}(t)$ *is a real periodic function of period* $T = 2\pi/\omega$, *L-integrable in* $[0, T]$ *with Fourier series*

$$f_{jh}(t) \sim \sum_{l=0}^{\infty} (a_{jhl} \cos l\omega t + b_{jhl} \sin l\omega t), \qquad j, h = 1, \ldots, n.$$

If either $a_{110} \neq 0$, $\varepsilon a_{110} < 0$, $M_h < 0$, $h = 2, \ldots, k$, *or* $a_{110} = 0$, $M_h < 0$, $h = 1, 2, \ldots, k$, $\varepsilon \neq 0$, *then for* $|\varepsilon|$ *sufficiently small all solutions of* (3.2.10) *approach zero as* $t \to +\infty$.

If either $a_{110} \neq 0$ *and we have at least one of the inequalities* $\varepsilon a_{110} < 0$, $M_h > 0$, $h = 2, \ldots, k$, *or* $a_{110} = 0$ *and we have at least one of the inequalities* $M_h > 0$, $h = 1, 2, \ldots, k$, *then for* $|\varepsilon| \neq 0$ *sufficiently small infinitely many solutions of* (3.2.10) *are unbounded in* $[0, +\infty)$.

In both cases M_h *is given by*

$$M_h = \sum_{\substack{j=1 \\ j \neq h}}^{n} \sum_{l=0}^{\infty} (p_{hjl} C_{hjl} + q_{hjl} D_{hjl}), \qquad h = 1, \ldots, k,$$

where

$p_{1j0} = \sigma_j^{-2}$, $j = 2, \ldots, k$; $p_{1j0} = d_j^{-1}$, $j = k+1, \ldots, n$; $q_{1j0} = 0$, $j = 2, \ldots, n$; $p_{1jl} = 2^{-1}(\sigma_j^2 - l^2\omega^2)^{-1}$, $q_{1jl} = 0$, $j = 2, \ldots, k$; $p_{1jl} = (d_j - l^2\omega^2)\, m_{jl}$, $q_{1jl} = c_j l \omega m_{jl}$, $m_{jl} = 2^{-1}[l^4\omega^4 + d_j^2 + l^2\omega^2(c_j^2 - 2d_j)]^{-1}$, $j = k+1, \ldots, n$; $p_{h10} = -2^{-1}\sigma_h^{-2}$, $q_{h10} = 0$, $h = 2, \ldots, k$; $p_{h1l} = 4^{-1}(-\sigma_h^2 + l^2\omega^2)^{-1}$, $q_{h1l} = 0$, $h = 2, \ldots, k$; $p_{hj0} = 0$, $q_{hj0} = 0$, $h = 2, \ldots, k$, $j = 2, \ldots, k$; $p_{hj0} = -2^{-1} c_j[(d_j - \sigma_h^2)^2 + \sigma_h^2 c_j^2]^{-1}$, $q_{hj0} = 0$, $h = 2, \ldots, k$, $j = k+1, \ldots, n$.

The formulas for p_{hjl}, q_{hjl}, $h = 2, \ldots, k$, $j = 2, \ldots, n$, $j \neq h$, $l = 1, 2, \ldots$, *are the same as those for* p_{hjl} *and* q_{hjl} *of Theorem* (3.2.iv) *above if we put* $c_j = 0$, $d_j = \sigma_j^2$, $j = 2, \ldots, k$.

(3.2.vi) *Consider the system*

$$\begin{aligned} & y_1'' + a\, y_1' + \varepsilon \sum_{h=1}^{n} f_{1h}(t)\, y_h = 0, && a > 0, \\ (3.2.11) \qquad & y_j'' + \sigma_j^2\, y_j + \varepsilon \sum_{h=1}^{n} f_{jh}(t)\, y_h = 0, && \sigma_j > 0, \quad j = 2, \ldots, k, \\ & y_j'' + c_j\, y_j' + d_j\, y_j + \varepsilon \sum_{h=1}^{n} f_{jh}(t)\, y_h = 0, && c_j, \quad d_j > 0, \quad j = k+1, \ldots, n, \end{aligned}$$

where $2 \leq k \leq m$, $2\sigma_j$, $\sigma_j + \sigma_h$, $\sigma_j - \sigma_h \not\equiv 0 \pmod{\omega}$, $j \neq h$, $j, h = 2, \ldots, k$, ε *real*, $f_{jh}(t)$ *is a real periodic function of period* $T = 2\pi/\omega$, *L-integrable in* $[0, T]$, *with Fourier series*

$$f_{jh}(t) \sim \sum_{l=0}^{\infty} (a_{jhl} \cos l\omega t + b_{jhl} \sin l\omega t), \qquad j, h = 1, \ldots, n.$$

If either $a_{110} \neq 0$, $\varepsilon a_{110} > 0$, $M_h < 0$, $h = 2, \ldots, k$, *or* $a_{110} = 0$, $M_h < 0$, $h = 1, 2, \ldots, k$, $\varepsilon \neq 0$, *then for* $|\varepsilon|$ *sufficiently small all solutions of* (3.2.11) *approach zero as* $t \to +\infty$. *If either* $a_{110} \neq 0$ *and we have at least one of the inequalities* $\varepsilon a_{110} > 0$, $M_h > 0$, $h = 2, \ldots, k$, *or* $a_{110} = 0$ *and we have at least one of the inequalities* $M_h > 0$, $h = 1, 2, \ldots, k$, *then for* $|\varepsilon| \neq 0$ *sufficiently small infinitely many solutions of* (3.2.11) *are unbounded in* $[0, +\infty)$. *In both cases* M_h *is given by*

$$M_h = \sum_{j=1}^{n} \sum_{l=0}^{\infty} p_{hjl} C_{hjl} + q_{hjl} D_{hjl}, \qquad h = 1, \ldots, k,$$

$p_{110} = -a^{-3}$, $q_{110} = 0$, $p_{1j0} = a^{-1}\sigma_j^{-2}$, $q_{1j0} = 0$, $j = 2, \ldots, k$;

$p_{1j0} = a^{-1} d_j^{-1}$, $q_{1j0} = 0$, $j = k+1, \ldots, n$;

$p_{11l} = -(2a)^{-1}(a^2 + l^2\omega^2)^{-1}$, $q_{11l} = 0$,

$p_{1jl} = (2a)^{-1}(\sigma_j^2 - l^2\omega^2)^{-1}$, $q_{1jl} = 0$, $j = 2, \ldots, k$, $l = 1, 2, \ldots$;

$p_{1jl} = (2a)^{-1}(d_j - l^2\omega^2)\, m_{jl}$, $q_{1jl} = (2a)^{-1} l\omega c_j m_{jl}$,

$m_{jl} = [d_j^2 + l^2\omega^2(c_j^2 - 2d_j) + l^4\omega^4]^{-1}$, $j = k+1, \ldots, n$, $l = 1, 2, \ldots$;

$p_{h10} = (2a)^{-1} a^2 (a^2 + \sigma_h^2)^{-1} \sigma_h^{-2}$, $q_{h10} = 0$,

$p_{h1l} = (4a)^{-1}(\sigma_h^2 + l^2\omega^2)^{-1}$, $q_{h1l} = 0$, $h = 2, \ldots, k$, $l = 1, 2, \ldots$;

$p_{hj0} = q_{hj0} = 0$, $j = 2, \ldots, k$, $h = 2, \ldots, k$;

$p_{hj0} = -2^{-1} c_j[(d_j - \sigma_h^2)^2 + \sigma_h^2 c_j^2]^{-1}$, $q_{hj0} = 0$, $j = k+1, \ldots, n$, $h = 2, \ldots, k$;

the formulas for p_{hjl}, q_{hjl}, $j=2,\ldots,n$, $h=2,\ldots,k$, $l=1,2,\ldots$, *are the same as the formulas for* p_{hjl}, q_{hjl} *of Theorem* (3.2.iv) *above if we put* $c_j=0$, $d_j=\sigma_j^2$, $j=2,\ldots,k$.

The proofs of (3.2.iv), (3.2.v), (3.2.vi) are omitted since they are obvious modifications of the previous ones.

Examples. Consider the system

$$\begin{aligned} y_1''+y_1+2\varepsilon\cos t\cdot y_1+\varepsilon\cos t\cdot y_2+\varepsilon\cos t\cdot y_3&=0\\ y_2''+2y_2+\varepsilon\cos t\cdot y_1+\varepsilon\cos t\cdot y_2+\varepsilon\cos t\cdot y_3&=0\\ y_3''+y_3'+y_3+\varepsilon\cos t\cdot y_1+\varepsilon\cos t\cdot y_2+\varepsilon\cos t\cdot y_3&=0 \end{aligned} \tag{3.2.12}$$

where $\varepsilon\neq 0$ is real and is thought of as being sufficiently small. By (3.2.vi) with $k=2$, $n=3$, $a_{110}=0$ we have $M_1=-0.01<0$, $M_2=-0.052<0$, and all solutions of (3.2.12) approach zero as $t\to+\infty$. By replacing in (3.2.12) f_{13}, or f_{31}, by $-\cos t$, we have $M_1=+0.01>0$. By replacing in (3.2.12) f_{32}, or f_{23}, by $-5\cos t$ we have $M_2=0.26>0$. By replacing in (3.2.12) f_{12} by $-2\sin t$, or f_{21} by $+2\sin t$, we have $M_1=0.24>0$. In all these cases infinitely many solutions are unbounded in $[0,+\infty)$, no matter how small $\varepsilon>0$ is chosen.

Consider the system

$$\begin{aligned} y_1''+2y_1+\varepsilon\sin t\cdot y_1+\varepsilon\cos t\cdot y_2&=0\\ y_2''+2y_2'+y_2+\varepsilon\cos t\cdot y_1+\varepsilon\cos t\cdot y_2&=0 \end{aligned} \tag{3.2.13}$$

where $\varepsilon\neq 0$ is real and is thought of as being sufficiently small. By (3.2.iv) with $k=1$, $n=2$, we have $M_1=-\frac{1}{16}<0$, and all solutions of (3.2.13) approach zero as $t\to+\infty$. By replacing in (3.2.13) $f_{21}=\cos t$ by $f_{21}=\sin t$, we have $M_1=+\frac{1}{16}>0$. By replacing in (3.2.13) f_{12} by $-\sin t$, we have $M_1=+\frac{1}{16}$. In both cases infinitely many solutions are unbounded in $[0,+\infty)$, no matter how small $\varepsilon\neq 0$ is chosen.

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Bounds for Solutions of Nonlinear Differential Systems

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1. Introduction

Bounds have been developed for solutions of differential systems of the form

$$\frac{dx}{dt} = A(t)\,x + F\big(t, x(t)\big) \tag{1.1}$$

by many authors $\big($[3], [9], [10], [11], [12], [13]; for more complete bibliography see [4], [5a], [6]$\big)$, usually based on the assumption that the nonlinear terms, represented by $F\big(t, x(t)\big)$ in (1.1), decrease faster than the linear terms as $\|x\| \to 0$ and have the effect of small perturbations of the solution to the linear part. In this article bounds are established for solutions of systems (1.1) with various kinds of nonlinearities, some that increase more slowly than the linear terms when $\|x\| \to \infty$ and others that decrease faster than the linear terms when $\|x\| \to 0$. In each case the bounds are derived from bounds to the general solution of the linear part assumed as known. In the case of (relatively) slowly increasing nonlinear terms the bounds hold for solutions with large initial values, in the case of (relatively) rapidly decreasing nonlinear terms the bounds are valid for solutions with small initial values. The bounds reflect at least the growth of the various terms involved and are rather sharp for some equations. They are developed from the solution to a certain nonlinear integral inequality [equation (2.6)]. This solution [see (2.17) and (3.18)] is an extension of a result for the corresponding linear integral inequality, which has been used frequently for the purpose of estimating the growth of solutions of differential systems and was first formulated as an explicit lemma by R. BELLMAN $\big($[1]; see also [4], p. 35, [7], [14]$\big)$.

2. Large Initial Values

The systems considered are of the form

$$\frac{dx}{dt} = A(t)\,x + F\big(t, x(t)\big) \tag{2.1}$$

where x and F are vectors with n real components and A is a matrix with n^2 real components. The norms used are denoted as $\|x\|$ and $\|A\|$ and are defined as the sums of the absolute values of the components. The components of $A(t)$ and $F(t, x)$ are defined for all $t \geqq t_0$ and all real x, locally L-integrable functions of t for fixed x and continuous in x for almost all $t \geqq t_0$. Moreover, it is assumed that

$$\|F(t, x)\| \leqq a(t)\,\varphi(\|x\|), \qquad t \geqq t_0 \tag{2.2}$$

where $a(t)$ is an integrable function, $\varphi(u)$ defined for $u>0$ as a positive non-decreasing continuous function which is submultiplicative, that is

$$\varphi(uv) \leqq \varphi(u)\,\varphi(v). \tag{2.3}$$

These assumptions will be made throughout this article and will not be restated.

For any real vector x_0 equation (2.1) has a solution $x=x(t)$ which exists on some interval to the right of $t=t_0$ and assumes the value x_0 at $t=t_0$. This solution satisfies the integral equation

$$x(t) = X(t)\,x_0 + \int_{t_0}^{t} X(t,\tau)\,F(\tau, x(\tau))\,d\tau, \tag{2.4}$$

where $X(t,\tau)$ is the matrix that solves the equation

$$\frac{dX}{dt} = A(t)\,X$$

and for which $X(\tau,\tau)$ is the identity matrix; $X(t)$ is an abbreviation for $X(t, t_0)$. Another assumption to be added to those stated above is

$$\|X(t,\tau)\| \leqq g(t)\,b(\tau), \qquad t_0 \leqq \tau \leqq t \tag{2.5a}$$

where $g(t)$, $b(t)$ are locally L-integrable functions and

$$\|X(t)\| \leqq g(t), \qquad b(t) \leqq \|X^{-1}(t)\|, \qquad t_0 \leqq t. \tag{2.5b}$$

Since $X(t,\tau) = X(t)\,X^{-1}(\tau)$, inequality (2.5) is always satisfied for $g(t) = \|X(t)\|$, $b(t) = \|X^{-1}(t)\|$. However, sharper bounds are often found. For example, if the matrix $A(t)$ in (2.1) is constant and has the n characteristic numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ then $\|X(t,\tau)\| = \|X(t-\tau)\| \leqq c(\alpha)\exp\alpha(t-\tau)$ for any $\alpha > \max \operatorname{Re}\lambda_i$ so that we may choose $g(t) = c(\alpha)\exp(\alpha t)$, $b(t) = \exp(-\alpha t)$. However, in Sections 2 and 3 we use $g(t)$ and $\|X(t)\|$ interchangeably, with the understanding that where $g(t)$ is identified with $\|X(t)\|$ there $b(t)$ is identified with $\|X^{-1}(t)\|$.

From (2.2), (2.4), (2.5) one derives the inequality

$$f(t) \leqq g(t)\left[f_0 + \int_{t_0}^{t} h(\tau)\,\varphi(f(\tau))\,d\tau\right], \tag{2.6}$$

which is satisfied in a certain interval to the right of $t=t_0$. The following abbreviations are used here:

$$f(t) = \|x(t)\|, \qquad f_0 = \|x_0\|, \tag{2.7a}$$

$$h(t) = a(t)\,b(t). \tag{2.7b}$$

If we put

$$r(t) = \int_{t_0}^{t} h(\tau)\,\varphi(f(\tau))\,d\tau, \tag{2.8}$$

then by (2.6) and (2.3)

$$\frac{dr}{dt} = h(t)\,\varphi(f(t)) \leqq h(t)\,\varphi[g(t)\,(f_0 + r(t))] \leqq h(t)\,\varphi(g(t))\,\varphi(f_0 + r(t)). \tag{2.9}$$

Thus, if $r' = dr/dt$,

$$\frac{r'}{\varphi(f_0 + r)} \leq h\,\varphi(g). \tag{2.10}$$

Let $\psi(x)$ be an integral of $\varphi(u)$ defined by

$$\psi(x) = \int_1^x \frac{1}{\varphi(u)}\,du, \qquad x > 0. \tag{2.11}$$

Then $\psi(x) > 0$ for $x > 1$, ψ is continuous and strictly increasing. Let $x = \psi^*(y)$ denote the inverse function defined for $\psi(0) < y < \psi(\infty)$. Because of the submultiplicative property (2.3), $\psi(x)$ satisfies the inequality

$$\psi(x+y) - \psi(x) = \int_x^{x+y} \frac{1}{\varphi(u)}\,du = x \int_1^{1+y/x} \frac{1}{\varphi(xu)}\,du \geqq \frac{x}{\varphi(x)}\,\psi\!\left(1 + \frac{y}{x}\right), \quad x, y > 0. \tag{2.12}$$

Let inequality (2.10) be integrated between the limits t_0 and t. Since $r(t_0) = 0$ one obtains

$$\psi(f_0 + r(t)) - \psi(f_0) \leqq s(t), \tag{2.13}$$

where the abbreviation

$$s(t) = \int_{t_0}^{t} h(\tau)\,\varphi(g(\tau))\,d\tau = \int_{t_0}^{t} a(\tau)\,b(\tau)\,\varphi(\|X(\tau)\|)\,d\tau \tag{2.14}$$

is used. Using (2.12) in (2.13) one obtains

$$\psi\!\left(1 + \frac{r(t)}{f_0}\right) \leq \frac{\varphi(f_0)}{f_0}\,s(t),$$

and if

$$\frac{\varphi(f_0)}{f_0}\,s(t) < \psi(\infty) \tag{2.15}$$

then

$$1 + \frac{r(t)}{f_0} \leqq \psi^*\!\left(\frac{\varphi(f_0)}{f_0}\,s(t)\right), \tag{2.16}$$

and because of (2.6)

$$f(t) \leqq f_0\,\psi^*\!\left(\frac{\varphi(f_0)}{f_0}\,s(t)\right) g(t). \tag{2.17}$$

For the solution $x(t)$ of (2.1) this means

$$\|x(t)\| \leq \|x_0\|\,\psi^*\!\left(\frac{\varphi(\|x_0\|)}{\|x_0\|}\,s(t)\right) \|X(t)\|. \tag{2.18}$$

If condition (2.15) is satisfied then the right-hand term of (2.18) gives a finite upper bound for the solution x in the interval from t_0 to t. It then follows that the solution can be extended beyond t. Thus, the solution exists in any interval $t_0 \leqq t \leqq T$ for which

$$s(T) < \frac{\|x_0\|}{\varphi(\|x_0\|)}\,\psi(\infty) = \frac{\|x_0\|}{\varphi(\|x_0\|)} \int_1^{\infty} \frac{du}{\varphi(u)}. \tag{2.19}$$

It should be noticed that if $u/\varphi(u)$ increases indefinitely with u then T increases indefinitely with $\|x_0\|$, that is the solution exists in an arbitrarily large interval $t_0 \leqq t \leqq T$ if only $\|x_0\|$ is sufficiently large.

If the integral $\int_1^\infty du/\varphi(u)$ is divergent, the solution exists for all $t \geqq t_0$ and any $x_0 \neq 0$. In the interval $t_0 \leqq t \leqq T$, the right-hand term of (2.18) represents an upper bound for $\|x(t)\|$. In particular if the integral $\int_1^\infty du/\varphi(u)$ is divergent and the integral

$$s(\infty) = \int_0^\infty a(\tau)\, b(\tau)\, \varphi(\|X(\tau)\|)\, d\tau \tag{2.20}$$

is convergent, then $\|x(t)\|$ is bounded on $t_0 \leqq t < \infty$ by $K\|X(t)\|$, where K depends on $\|x_0\|$, but not on t. Thus, if $\|X(t)\|$ is bounded, that is, if the linear system $x' = Ax$ is stable, then every solution of (2.1) (with $x_0 \neq 0$) is bounded, and if $x' = Ax$ is asymptotically stable, then every solution of (2.1) tends to 0 as $t \to \infty$. These results are summarized in

Theorem 2.1. *The equation*

$$\frac{dx}{dt} = A(t)\, x + F(t, x)$$

has a solution $x = x(t)$ with $x(t_0) = x_0 \neq 0$ in any interval $t_0 \leqq t \leqq T$ for which

$$s(T) = \int_0^T a(\tau)\, b(\tau)\, \varphi(\|X(\tau)\|)\, d\tau < \frac{\|x_0\|}{\varphi(\|x_0\|)} \int_1^\infty \frac{du}{\varphi(u)}$$

and an upper bound is given by

$$\|x(t)\| \leqq \|x_0\|\, \psi^*\left(\frac{\varphi(\|x_0\|)}{\|x_0\|}\, s(t)\right) \|X(t)\|, \qquad t_0 \leqq t \leqq T.$$

If the integral $\int_1^\infty du/\varphi(u)$ is divergent, then the solution exists for all $t \geqq t_0$ and any $x_0 \neq 0$. If moreover the integral $s(\infty)$ is convergent, then $\|x(t)\|/\|X(t)\|$ is bounded (on $t \geqq t_0$) and every solution $x(t)$ is bounded if the system $x' = Ax$ is stable, and tends to 0 as $t \to \infty$ if $x' = Ax$ is asymptotically stable.

For illustration, assume

$$\|F(t, x)\| \leqq a(t)\, \|x\|^{1-k}, \qquad 0 \leqq k \leqq 1. \tag{2.21}$$

Then the integral $\int_1^\infty du/\varphi(u) = \int_1^\infty u^{k-1}\, du$ is divergent, and equation (2.1) has a solution $x(t)$ existing for all $t \geqq t_0$ for any $x_0 = x(t_0) \neq 0$. Moreover

$$\|x(t)\| \leqq \left(\|x_0\|^k + k\, s(t)\right)^{1/k} \|X(t)\|, \qquad t \geqq t_0 \tag{2.22}$$

if $k \neq 0$ and

$$\|x(t)\| \leqq \|x_0\|\, e^{s(t)}\, \|X(t)\|, \qquad t \geqq t_0 \tag{2.23}$$

if $k = 0$. Here

$$s(t) = \int_{t_0}^t a(\tau)\, b(\tau)\, \|X(\tau)\|^{1-k}\, d\tau.$$

If $s(\infty) < \infty$ and the system $x' = Ax$ is stable (asymptotically stable), then the solution $x = 0$ of (2.1) is stable (asymptotically stable) if $k = 0$, but need not be so if $k \neq 0$. This latter occurrence is illustrated by the single equation

$$\frac{dx}{dt} = -x + e^{-t}|x|^{\frac{1}{2}} \operatorname{sgn} x, \qquad t > 0 \tag{2.24}$$

whose solution for $x(0) = x_0 \neq 0$ is

$$x(t) = e^{-t}\left(1 + |x_0|^{\frac{1}{2}} - e^{-\frac{1}{2}t}\right)^2 \operatorname{sgn} x_0. \tag{2.25}$$

It is bounded, but the solution $x = 0$ is not stable. The general bound (2.22) gives in this special case

$$|x(t)| \leqq e^{-t}\left(1 + |x_0|^{\frac{1}{2}} - e^{-\frac{1}{2}t}\right)^2,$$

which is the exact least upper bound.

It was seen that if $\int_1^\infty du/\varphi(u) = \infty$ then the solution $x = x(t; x_0)$ of (2.1) exists for all $t \geqq t_0$, but does not necessarily tend to 0 as $\|x_0\| \to 0$. However $x(t; x_0)$ remains bounded (for fixed t) as $\|x_0\| \to 0$ provided $\varphi(u)$ satisfies additional conditions. To prove this, it is sufficient, by (2.18), to show that

$$y(x) = x\psi^*\left(\frac{\varphi(x)}{x} s\right), \qquad x > 0 \tag{2.26}$$

is bounded near $x = 0$, for any fixed s. This is obviously the case if

$$\varphi(x) \leqq Ax, \qquad 0 \leqq x \leqq 1 \tag{2.27}$$

where A is a constant. (2.26) is also bounded if

$$\left.\begin{aligned} &\varphi(x) \leqq Ax^\alpha, && 0 \leqq x \leqq 1 \\ &\int_1^x \frac{du}{\varphi(u)} \geqq B(x^\beta - 1), && 1 \leqq x < \infty \end{aligned}\right\} \tag{2.28}$$

where $A, B > 0, \alpha, \beta$ are constants and

$$\beta > 0, \qquad \alpha + \beta \geqq 1. \tag{2.29}$$

For then

$$B\left[\left(\frac{y}{x}\right)^\beta - 1\right] \leqq \psi\left(\frac{y}{x}\right) \leqq \frac{\varphi(x)}{x} s \leqq A s x^{\alpha-1}$$

or

$$y^\beta \leqq \frac{A}{B} s\left(1 + x^{\alpha-1}\right) x^\beta, \qquad 0 < x \leqq 1 \tag{2.30}$$

which is bounded because of (2.29).

Moreover, it is seen that $\lim\limits_{x \to 0} y(x) = 0$ if (2.27) is satisfied and also if (2.28), (2.29) with $\alpha + \beta > 1$ are satisfied. Thus we have proved

Theorem 2.2. *If either* (a) $x^{-1}\varphi(x)$ *is bounded near* $x = 0$ *and* $\left[\int_1^x du/\varphi(u)\right]^{-1}$ *is bounded near* $x = \infty$ *or* (b) $x^{-\alpha}\varphi(x)$ *is bounded near* $x = 0$, $x^\beta\left[\int_1^x du/\varphi(u)\right]^{-1}$ *is bounded near* $x = \infty$ *and* $\beta > 0$, $\alpha + \beta \geqq 1$, *then every solution* $x = x(t; x_0)$ *of* (2.1) *with* $x_0 = x(t_0) \neq 0$ *can be extended to the interval* $t \geqq t_0$ *and* $\|x(t; x_0)\|$ *is a bounded function of* x_0 *near* $x_0 = 0$. *Moreover,* $\|x(t; x_0)\| \to 0$ *as* $\|x_0\| \to 0$ *if either* (a) *is satisfied or* (b) *is with* $\alpha + \beta > 1$.

3. Small Initial Values

The system considered in this section is again

$$\frac{dx}{dt} = A(t)\,x + F\big(t, x(t)\big). \tag{3.1}$$

The same general assumptions on A, F are made as in Section 2; in particular, inequalities (2.2), (2.3) and (2.5) are assumed to hold. In addition we assume

$$\int_0^a \frac{du}{\varphi(u)} = \infty \tag{3.2}$$

for any positive a. We consider solutions $x = x(t)$ of (3.1) for which $x_0 = x(t_0)$ is sufficiently small. The symbols $X(t)$, $b(t)$, $f(t)$, $a(t)$, $g(t)$, $h(t)$, $r(t)$, $s(t)$, $\varphi(t)$, $\psi(t)$ are to have the same meaning as in Section 2.

We introduce numbers R, T as follows. If $\psi(\infty) = \int_1^\infty du/\varphi(u) < \infty$, then we put $R = \infty$, $T = \infty$. Otherwise we choose, for given $\|x_0\| > 0$, numbers R, T such that

$$t_0 < T, \qquad \|x_0\|\,\psi^*\Big(\frac{\varphi(\|x_0\|)}{\|x_0\|}\,s(T)\Big) < R. \tag{3.3}$$

Let the function $\chi(x)$, $0 \leqq x < R$, be defined as

$$\chi(0) = 0, \qquad \chi(x) = 1\Big/\int_x^R \frac{du}{\varphi(u)}, \qquad 0 < x < R. \tag{3.4}$$

This function is continuous and strictly increasing, positive for $0 < x < R$, and the inverse function $\chi^*(y)$ is defined for $0 \leqq y < \infty$. By (2.16) and (3.3)

$$\|x_0\| + r(t) \leqq \|x_0\|\,\psi^*\Big(\frac{\varphi(\|x_0\|)}{\|x_0\|}\,s(t)\Big) < R, \qquad t_0 \leqq t < T. \tag{3.5}$$

Thus the differential inequality (2.10) may be integrated between t_0 and t with the use of the integral $1/\chi$ of $1/\varphi$. One obtains

$$1/\chi(\|x_0\|) - 1/\chi(\|x_0\| + r(t)) \leqq s(t), \qquad t_0 \leqq t < T$$

and

$$\chi(\|x_0\| + r(t)) \leqq \frac{\chi(\|x_0\|)}{1 - \chi(\|x_0\|)\,s(t)}, \qquad t_0 \leqq t < T \tag{3.6}$$

provided

$$\chi(\|x_0\|)\,s(t) < 1, \qquad t_0 \leqq t < T. \tag{3.7}$$

If $\int_1^\infty du/\varphi(u) < \infty$, then $R = \infty$ and (3.7) is a condition on the smallness of $\|x_0\|$. If $\int_1^\infty du/\varphi(u) = \infty$ then $\chi(x) \to 0$ as $R \to \infty$ and (3.7) is a condition on the size of R, no restriction on $\|x_0\|$ being necessary.

From (3.6) and (2.6) we have

$$\|x(t)\| \leqq \chi^*\Big(\frac{\chi(\|x_0\|)}{1 - \chi(\|x_0\|)\,s(t)}\Big)\,\|X(t)\|, \qquad t_0 \leqq t < T. \tag{3.8}$$

Since T was arbitrary and since conditions (3.3) and (3.7) can be satisfied with positive $\|x_0\|$ it follows that equation (3.1) has solutions $x(t) \neq 0$ which exist in arbitrarily large intervals $t_0 \leqq t < T$ and for which the estimate (3.8) is valid. These results are summarized in

Theorem 3.1. *If the integral* $\int_0^a du/\varphi(u)$ $(a>0)$ *is divergent, then the equation*

$$\frac{dx}{dt} = A(t)\,x + F(t, x(t))$$

has a solution $x = x(t)$ *in the arbitrarily large interval* $t_0 \leqq t < T$, *with the upper bound*

$$\|x(T)\| \leqq \chi^*\left(\frac{\chi(\|x_0\|)}{1-\chi(\|x_0\|)\,s(t)}\right)\|X(t)\|, \qquad t_0 \leqq t < T$$

provided $x(t_0) = x_0$ *is so small that*

$$s(T) = \int_0^T a(\tau)\,b(\tau)\,\varphi(\|X(\tau)\|)\,d\tau < \int_{\|x_0\|}^R \frac{du}{\varphi(u)}.$$

Here $\chi(x) = 1\Big/\int_x^R du/\varphi(u)$, *and* R *is any positive number*

$$> \|x_0\|\,\psi^*(\varphi(\|x_0\|)\,s(T)/\|x_0\|), \quad \textit{or} \quad R = \infty \quad \textit{if} \quad \int_1^\infty du/\varphi(u) < \infty.$$

For example, if

$$\|F(t,x)\| \leqq a(t)\,\|x\|^{1+k}, \qquad k>0, \tag{3.9}$$

then equation (3.1) has a solution $x = x(t)$ in $t_0 \leqq t < T$, with the upper bound

$$\|x(t)\| \leqq \frac{\|x_0\|}{[1-k\|x_0\|^k s(t)]^{1/k}}\,\|X(t)\|, \qquad t_0 \leqq t < T \tag{3.10}$$

if

$$k\|x_0\|^k s(T) = k\|x_0\|^k \int_0^T a(\tau)\,b(\tau)\,\|X(\tau)\|^{1+k}\,d\tau < 1. \tag{3.11}$$

If the integral $s(\infty) < \infty$ then we can take $\|x_0\| > 0$ small enough so that the solution $x = x(t; x_0)$ exists and estimate (3.8) holds for all $t \geqq t_0$. Since $\chi(x) \to 0$ as $x \to 0$ it follows in this case that $\|x(t; x_0)\|/\|X(t)\| \to 0$ as $\|x_0\| \to 0$ uniformly for $t \geqq t_0$. Thus, in this case, $x = 0$ is a stable (asymptotically stable) solution of (3.1) if the system $x' = Ax$ is stable (asymptotically stable). Again assuming $s(\infty) < \infty$, a more precise statement on the order of $\sup_{t \geqq t_0} \|x(t; x_0)\|/\|X(t)\|$ as $\|x_0\| \to 0$ can be made for the case where $\varphi(u_0) < u_0$ for some u_0 (necessarily $0 < u_0 < 1$). By (2.3) for $0 \leqq x < R$, $0 < y \leqq 1$,

$$\begin{aligned}
\chi(xy) &= 1\Big/\int_{xy}^R \frac{du}{\varphi(u)} \\
&= 1\Big/y\int_x^{R/y} \frac{du}{\varphi(yu)} \\
&\leqq \varphi(y)\Big/y\int_x^R \frac{du}{\varphi(u)},
\end{aligned}$$

that is

$$\chi(xy) \leqq \frac{\varphi(y)}{y}\chi(x), \qquad 0 \leqq x < R, \qquad 0 < y \leqq 1. \tag{3.12}$$

It follows that

$$\chi^*\left(\frac{y}{\varphi(y)}\chi(x)\right) \leqq \frac{x}{y}, \qquad 0 \leqq x < R, \qquad 0 < y \leqq 1. \tag{3.13}$$

The function $\varphi(y)/y$ has by (2.3) a value $\geqq 1$ for $y=1$, and by assumption a value $v_0 < 1$ for $y = u_0$. Thus $v = \varphi(y)/y$ takes on every value between v_0 and 1 as y varies from u_0 to 1. Some inverse function is defined for $v_0 \leqq v \leqq 1$, and we have $u_0 \leqq y < 1$. Hence, the equation

$$\frac{\varphi(y)}{y} = 1 - \chi(\|x_0\|)\, s(\infty) \tag{3.14}$$

has a (not necessarily continuous) solution $y = 1/\omega(\|x_0\|)$ with $\omega(\|x_0\|) \leqq 1/u_0$ if $\|x_0\|$ is so small that

$$\chi(\|x_0\|)\, s(\infty) \leqq 1 - v_0 = 1 - \frac{\varphi(u_0)}{u_0}. \tag{3.15}$$

Using (3.13) we then have

$$\chi^*\left(\frac{\chi(\|x_0\|)}{1 - \chi(\|x_0\|)\, s(\infty)}\right) \leqq \omega(\|x_0\|)\, \|x_0\|, \tag{3.16}$$

and by (3.8)

$$\sup_{t \geqq t_0} \frac{\|x(t; x_0)\|}{\|X(t)\|} \leqq \omega(\|x_0\|)\, \|x_0\| \tag{3.17}$$

if $\|x_0\|$ satisfies condition (3.15). We thus have proved

Theorem 3.2. *If the integral* $\int_0^a du/\varphi(u)$ $(a>0)$ *is divergent and the integral* $\int_0^\infty a(\tau)\, b(\tau)\, \varphi(\|X(\tau)\|)\, d\tau$ *is convergent, then there exists a solution* $x = x(t; x_0)$ *of* (3.1) *on the whole interval* $t \geqq t_0$ *for all sufficiently small* $\|x_0\|$, *and*

$$\sup_{t \geqq t_0} \frac{\|x(t; x_0)\|}{\|X(t)\|} = O(1) \quad \text{as} \quad \|x_0\| \to 0.$$

If moreover $\varphi(u_0) < u_0$ *for some* u_0, *then*

$$\sup_{t \geqq t_0} \frac{\|x(t; x_0)\|}{\|X(t)\|} = O(\|x_0\|) \quad \text{as} \quad \|x_0\| \to 0.$$

In these cases $x = 0$ *is a stable (asymptotically stable) solution of* (3.1) *if* $x' = Ax$ *is a stable (asymptotically stable) system.*

The following example may serve to illustrate this theorem. The system

$$\begin{aligned} \frac{dx_1}{dt} &= -a\, x_1 \\ \frac{dx_2}{dt} &= (\sin\log t + \cos\log t - 2a)\, x_2 + x_1^{1+k}, \qquad k > 0, \quad t > 0 \end{aligned} \tag{3.18}$$

is used (with $k=1$) by O. PERRON ([12]; see also [4], p. 87) as a counter example for a proposed stability theorem. The solution of (3.18) is

$$\begin{aligned} x_1 &= c_1 e^{-at} \\ x_2 &= e^{t \sin \log t - 2at} \left[c_2 + c_1^{1+k} \int_0^t e^{-\tau \sin \log \tau + a(1-k)\tau} \, d\tau \right], \end{aligned} \tag{3.19}$$

where $c_1 = x_1(0)$, $c_2 = x_2(0)$. Here

$$\|X(t)\| = e^{-at} + e^{t \sin \log t - 2at},$$

$$\|X^{-1}(t)\| = e^{at} + e^{-t \sin \log t + 2at}$$

and the integral

$$s(\infty) = \int_0^\infty (e^{-a\tau} + e^{\tau \sin \log \tau - 2a\tau})^{1+k} (e^{a\tau} + e^{-\tau \sin \log \tau + 2a\tau}) \, d\tau \tag{3.20}$$

is convergent if either

$$\frac{1}{2} < a \leqq 1 \quad \text{and} \quad k > \frac{1}{a - \frac{1}{2}} \tag{3.21 a}$$

or

$$a \geqq 1 \quad \text{and} \quad k > 1 + \frac{1}{a}. \tag{3.21 b}$$

In either case, application of Theorem 3.2 reveals, and the explicit expression (3.19) confirms, that the solution $x=0$ of system (3.18) is asymptotically stable and

$$|x_1(t)| + |x_2(t)| \leqq C(|c_1| + |c_2|)(e^{-at} + e^{t \sin \log t - 2at}), \qquad t \geqq 0$$

for some constant C independent of t and c_1, c_2. Although $x=0$ is not a stable solution of (3.18) if $k=1$ and a is sufficiently small (PERRON's example), it is an asymptotically stable solution for any $a > \frac{1}{2}$ if only k is sufficiently large.

4. Almost Constant A

More specific results are obtained for the case where the matrix $A(t)$ of the system

$$\frac{dx}{dt} = A(t)\, x + F(t, x(t)) \tag{4.1}$$

is almost constant. By this we shall mean the following: There exist complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ [the "asymptotic characteristic values" of $A(t)$] and matrices $S(t,\tau)$, $B(t,\tau)$ such that $\|S(t,\tau)\|$, $\|S^{-1}(t,\tau)\|$, $\|B(t,\tau)\|$ are bounded for $t_0 \leqq \tau \leqq t < \infty$ and the fundamental solution matrix $X(t,\tau)$ of the system $x' = Ax$ is given by

$$S X S^{-1} = \Lambda(I + B) \tag{4.2}$$

where Λ is the diagonal matrix with the elements $\exp \lambda_k (t-\tau)$ $(k = 1, 2, \ldots, n)$ and I is the unit matrix. Some conditions which assure that the matrix $A(t)$ is almost constant according to this definition can be deduced from known results on asymptotic expansions of solutions to linear systems (see, e.g., [8], [11], [12]) and from a direct discussion (see, e.g., [2], [3], [4]). General conditions

have been given in connection with CESARI's method of reduction of a linear system to "L-diagonal form" (see, *e.g.*, [*5*], [*15*], and for more references [*5a*]).

From (4.2) one derives the bound

$$\|X(t,\tau)\| \leqq C\, e^{\alpha(t-\tau)}, \qquad t_0 \leqq \tau \leqq t, \tag{4.3}$$

where α is any real number $> \lambda = \max \operatorname{Re} \lambda_i$ $(i = 1, 2, \ldots, n)$ and C is independent of t, τ. Thus we may identify $b(t)$, $g(t)$ of Sections 2, 3 with $C e^{-\alpha t}$, $e^{\alpha t}$ respectively.

If now

$$\|F(t,x)\| \leqq a(t)\, \|x\|^k, \qquad t \geqq t_0, \quad k \geqq 0 \tag{4.4}$$

and we choose $\varphi(u) = u^k$ then by (2.13)

$$s(t) = C \int_{t_0}^{t} a(\tau)\, e^{(k-1)\alpha\tau}\, d\tau \tag{4.5}$$

and $s(\infty) < \infty$ if $a(t)$ has an exponential type number $< (1-k)\lambda$, that is $a(t) \exp[(k-1)\lambda + \varepsilon]\, t$ is bounded on $t \geqq t_0$ for some $\varepsilon > 0$. In this case Theorem 2.2 applies if $k \leqq 1$ and Theorem 3.1 if $k > 1$. The results are formulated in

Theorem 4.1. *Assume* $\|F(t,x)\| \leqq a(t)\, \|x\|^k$ $(k \geqq 0)$, $A(t)$ *is an almost constant matrix in the sense defined above with the asymptotic characteristic numbers* λ_1, $\lambda_2, \ldots, \lambda_n$, *and* $a(t)$ *has type number* $< (1-k)\lambda$ *where* $\lambda = \max \operatorname{Re} \lambda_i$. *Then the system*

$$\frac{dx}{dt} = A(t)\, x + F(t, x(t)), \qquad t \geqq t_0$$

has a solution $x = x(t; x_0)$ *existing for all* $t \geqq t_0$ *for any* $x_0 \neq 0$ *if* $k \leqq 1$ *and for all sufficiently small* x_0 *if* $k > 1$. *In either case the solution has type number* $\leqq \lambda$.

In the case $k = 1$ the theorem requires that $a(t)$ has a negative type number. However, weaker conditions would suffice. Since this case has been treated extensively (see, *e.g.*, [*3*], [*4*], [*5a*], [*6*], [*9*]), it is not investigated any further here.

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Coefficient Problems in Systems of Partial Differential Equations

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1. Introduction

Various coefficient problems have been considered for partial differential equations of the form

$$\tag{1.1} \tfrac{1}{4}[\Delta v + C(x, y)\, v] \equiv u_{z z^*} + c(z, z^*)\, u = 0$$

where $z = x + iy$, $z^* = x - iy$. That is, relations have been obtained connecting basic properties of a solution

$$\tag{1.2} u(z, z^*) = \sum_{m,n=0}^{\infty} u_{mn} z^m z^{*n}$$

of (1.1) with properties of the coefficients u_{mn}. These relations yield information about the domain of regularity, the location and nature of singularities, the growth of $u(z, z^*)$ and other properties. *Cf.* BERGMAN [*1, 4, 5*], INGERSOLL [*6*], KREYSZIG [*7*]*. For this purpose certain *Bergman operators* (see below) serve as a powerful tool. These operators, which transform analytic functions $f(z)$ of a complex variable into solutions of (1.1), preserve many properties of $f(z)$, so that theorems on analytic functions yield corresponding theorems on solutions of (1.1). These Bergman operators have the property that all information about $u(z, z^*)$ is obtained from the sequence (u_{m0}), $m = 0, 1, \ldots$, of the coefficients in (1.2). A recent investigation [*7*] studied the extent to which information about $u(z, z^*)$ can be obtained from other sequences of coefficients in (1.2).

The unification of methods is one of the aims of the theory of linear partial differential equations. The consideration of coefficient problems is an approach in this direction. Indeed, various relations between properties of $u(z, z^*)$ and the coefficients in (1.2) are independent of the coefficients of the partial differential equation. It is therefore important to extend these considerations to systems of partial differential equations.

In the present paper we shall investigate coefficient problems for solutions

$$\tag{1.3} u(z_1, z_1^*, z_2, z_2^*) = \sum_{m,n,p,q=0}^{\infty} u_{mnpq}\, z_1^m z_1^{*n} z_2^p z_2^{*q}$$

* *Cf.* the references at the end of this paper.

of systems of partial differential equations

$$\tfrac{1}{4}[\Delta v + C_k(x_k, y_k)\,v] \equiv u_{z_k z_k^*} + c_k(z_k, z_k^*)\,u = 0, \qquad k = 1, 2, \tag{1.4}$$

$$z_k = x_k + i\,y_k, \qquad z_k^* = x_k - i\,y_k,$$

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\,\frac{\partial}{\partial y_k}\right), \qquad \frac{\partial}{\partial z_k^*} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\,\frac{\partial}{\partial y_k}\right),$$

$$u(z_1, z_1^*, z_2, z_2^*) \equiv v(x_1, y_1, x_2, y_2), \qquad c_k(z_k, z_k^*) = \tfrac{1}{4}\,C_k(x_k, y_k),$$

whose coefficients $c_k(z_k, z_k^*)$ are entire functions.

BERGMAN & SCHIFFER [2] proved the existence of operators transforming pairs of analytic functions of two complex variables into solutions of the system (1.4). Coefficient problems for these solutions were considered by BERGMAN [3]. Those operators have the property that all information about $u(z_1, z_1^*, z_2, z_2^*)$ can be obtained from the sequences

$$(u_{m0p0}), \quad m, p = 0, 1, \ldots, \qquad \text{and} \qquad u(_{m00q}), \quad m, q = 0, 1, \ldots,$$

of the coefficients in (1.3). Hence arises the basic problem as to what extent information about the behavior of $u(z_1, z_1^*, z_2, z_2^*)$ can be obtained from other sequences of these coefficients, say, from the sequence

$$(u_{mnpq}), \quad n > 0, \quad q > 0, \quad \text{both fixed}, \quad m, p = 0, 1, \ldots.$$

This problem will be considered in the present paper.

2. Bergman Operators

We now consider some basic properties of the Bergman operators needed for what follows. For better understanding we mention also the case of a single partial differential equation (1.1). In this case, every solution regular at the origin can be represented in the form

$$u(z, z^*) = \int_{-1}^{1} E(z, z^*, t)\, f\left(\tfrac{1}{2}z[1 - t^2]\right)(1 - t^2)^{-\frac{1}{2}}\,dt. \tag{2.1}$$

This representation can be cast into the form

$$u(z, z^*) = g(z) + \sum_{n=1}^{\infty} Q_n(z, z^*) \int_0^z (z - \zeta)^{n-1} g(\zeta)\,d\zeta, \tag{2.2}$$

where

$$Q_n(z, 0) = 0, \qquad n = 1, 2, \ldots,$$

and therefore

$$u(z, 0) = g(z). \tag{2.3}$$

The *associated function* $g(z)$ of $u(z, z^*)$ is an analytic function of a complex variable, regular at the origin. The functions $Q_n(z, z^*)$ depend on $c(z, z^*)$ in (1.1) but not on $g(z)$. They are entire if $c(z, z^*)$ is entire. The relation (2.3) yields the starting point for considering coefficient problems. *Cf.* BERGMAN [1, 4, 5], INGERSOLL [6], KREYSZIG [7]. We note that

$$f(z) = \frac{\sqrt{2z}}{\Gamma(\frac{1}{2})}\,\frac{d^{\frac{1}{2}} g(2z)}{dz^{\frac{1}{2}}};$$

cf. (2.1) and (2.2).

When passing from a single partial differential equation to a system we encounter considerable complication. While in the case of one equation we apply the Bergman operator to analytic functions of *one* complex variable, in the case of a system we have to apply it to analytic functions of *two* complex variables, whose theory is not sufficiently developed for our purpose. In particular, little is known about relations between coefficients of the development of functions $f(z_1, z_2)$ and the location and type of singularities of these functions. Thus for studying systems of equations it is natural to investigate first solutions whose associated functions are so chosen that relations of the afore-mentioned type can be obtained. This will be done systematically in the subsequent sections.

Each solution of the system (1.4) which is regular at the origin can be represented in the form

$$(2.4)\qquad u(z_1, z_1^*, z_2, z_2^*) = \int_{-1}^{1}\int_{-1}^{1} E_1(z_1, z_1^*, t_1)\, E_2(z_2, z_2^*, t_2)\, \{f_1(z_1[1-t_1^2], z_2[1-t_2^2]) + f_2(z_1[1-t_1^2], z_2^*[1-t_2^2])\}\,(1-t_1^2)^{-\frac{1}{2}}(1-t_2^2)^{-\frac{1}{2}}\,dt_1\,dt_2.$$

The *associated functions* f_1 and f_2 of u are analytic functions of two complex variables, regular at the origin. The *generating functions* E_1 and E_2 are entire functions of the respective variables z_1, z_1^* and z_2, z_2^*, and regular functions of t_1 and t_2, respectively, for $|t_k| \leqq 1$, $k=1, 2$. Furthermore,

$$(2.5)\qquad E_k(z_k, 0, t_k) = 1, \qquad E_k(0, z_k^*, t_k) = 1, \qquad k = 1, 2.$$

E_1 and E_2 depend on c_1 and c_2 in (1.4) but not on the choice of f_1 and f_2. *Cf.* Bergman & Schiffer [*2*].

We note that if x_1, y_1, x_2, and y_2 are real then z_k and z_k^* are conjugate; otherwise z_k and z_k^* are independent.

From the complex solution $u(z_1, z_1^*, z_2, z_2^*)$ we may obtain another solution of (1.4) by applying to (2.4) the operator R defined as follows:

$$\mathrm{R}\left(\sum_{m,n=0}^{\infty} a_{mn}\, z_k^m\, z_k^{*n}\right) = \tfrac{1}{2}\left(\sum_{m,n=0}^{\infty} a_{mn}\, z_k^m\, z_k^{*n} + \sum_{m,n=0}^{\infty} \bar{a}_{mn}\, z_k^{*m}\, z_k^n\right).$$

The solution $U=\mathrm{R}(u)$ of (1.4) is real for real values of the variables x_k, y_k, $k=1, 2$; we shall call such a solution a *real solution*. Obviously, for real values of x_k, y_k the operator R is identical with the operator Re ("take the real part").

From (2.4) Bergman [*3*] derived a representation suitable for considering coefficient problems of $u(z_1, z_1^*, z_2, z_2^*)$; it has the property that the relation between the solutions of (1.4) and their associated functions becomes very simple. Bergman proved that for each system (1.4) there exist four functions $S_k(z_k, z_k^*, \zeta_k)$, $T_k(z_k, z_k^*, \zeta_k)$, $k=1, 2$, such that each complex solution of (1.4), which is regular at the origin can be represented in the form

$$\begin{aligned}(2.6)\qquad u(z_1, z_1^*, z_2, z_2^*) = P(g_1, g_2) &\equiv g_1(z_1, z_2) + g_2(z_1, z_2^*)\\ &+ \int_0^{z_1} S_1 g_1(\zeta_1, z_2)\,d\zeta_1 + \int_0^{z_2} S_2 g_1(z_1, \zeta_2)\,d\zeta_2\\ &+ \int_0^{z_1} T_1 g_2(\zeta_1, z_2^*)\,d\zeta_1 + \int_0^{z_2^*} T_2 g_2(z_1, \zeta_2)\,d\zeta_2\\ &+ \int_0^{z_1}\int_0^{z_2} S_1 S_2 g_1(\zeta_1, \zeta_2)\,d\zeta_1\,d\zeta_2 + \int_0^{z_1}\int_0^{z_2^*} T_1 T_2 g_2(\zeta_1, \zeta_2)\,d\zeta_1\,d\zeta_2,\end{aligned}$$

where the *associated functions* $g_1(z_1, z_2)$ and $g_2(z_1, z_2^*)$ of the solution u are analytic functions of two complex variables, regular at the origin.

For a real solution of (1.4) we then have

$$(2.7)\quad \begin{aligned} U(z_1, z_1^*, z_2, z_2^*) &= \mathrm{R}\left(P(g_1, g_2)\right) \\ &= \tfrac{1}{2}\left\{P\left(g_1(z_1, z_2), g_2(z_1, z_2^*)\right) + \overline{P}\left(\overline{g}_1(z_1^*, z_2^*), \overline{g}_2(z_1^*, z_2)\right)\right\}. \end{aligned}$$

In (2.6) and (2.7) the associated function $g_2(z_1, z_2^*)$ can always be chosen so that

$$(2.8)\qquad g_2(0, z_2^*) = g_2(z_1, 0) = 0$$

holds; *cf.* [*2*]. From (2.5) it follows that

$$S_1(z_1, 0, \zeta_1) = S_2(z_2, 0, \zeta_2) = 0, \qquad T_1(z_1, 0, \zeta_1) = T_2(0, z_2^*, \zeta_2) = 0.$$

Using (2.8) we thus obtain from (2.6) the important relations

$$(2.9)\qquad u(z_1, 0, z_2, 0) = g_1(z_1, z_2)$$

and

$$(2.10)\qquad u(z_1, 0, 0, z_2^*) = g_2(z_1, z_2^*) + g_1(z_1, 0).$$

In the case of a real solution we obtain from (2.7) and (2.8)

$$(2.11)\qquad U(z_1, 0, z_2, 0) = \tfrac{1}{2}\left(g_1(z_1, z_2) + \overline{g}_1(0, 0)\right)$$

and

$$(2.12)\qquad U(z_1, 0, 0, z_2^*) = \tfrac{1}{2}\left(g_2(z_1, z_2^*) + g_1(z_1, 0) + \overline{g}_1(0, z_2^*)\right).$$

Hence from the sequence (u_{m0p0}) or (u_{m00q}) of the coefficients in the development (1.3) we may obtain information about basic properties of the complex solution u of (1.4), *cf.* BERGMAN [*3*]. The investigation is much the same as for a single partial differential equation.

Clearly, we should be able to obtain information about the behavior of u also from other sequences (u_{mnpq}), $n>0$, $q>0$, both fixed, $m, p = 0, 1, \ldots$. We thus arrive at the problem stated at the end of the preceding section. To solve this problem it suffices to derive relations between an arbitrary fixed sequence (u_{mnpq}), $n>0$, $q>0$, $m, p = 0, 1, \ldots$, and the sequence (u_{m0p0}). In doing this we may concentrate on complex solutions, since the generalization of the subsequent results to the case of real solutions will be immediate.

3. The Functions $u_{nq}(z_1, z_2)$

We wish to derive relations between an arbitrary given sequence (u_{mnpq}), $n>0$, $q>0$, both fixed, and the sequence (u_{m0p0}) of the coefficients in the representation (1.3). Using these relations and (2.9) we can obtain information about the associated function $g_1(z_1, z_2)$ of the corresponding solution $u(z_1, z_1^*, z_2, z_2^*)$ of (1.4) and thus about the afore mentioned and other properties of this solution.

In order derive such relations we introduce the functions

$$(3.1)\qquad u_{nq}(z_1, z_2) = \sum_{m,p=0}^{\infty} u_{mnpq}\, z_1^m\, z_2^p, \qquad n, q = 0, 1, \ldots.$$

The coefficients of each of these developments constitute just one of the sequences above. Relations between these functions are thus equivalent to the desired relations between the sequences.

From (1.3) and (3.1) we have

$$(3.2)\qquad u(z_1, z_1^*, z_2, z_2^*) = \sum_{n,q=0}^{\infty} u_{nq}(z_1, z_2)\, z_1^{*n}\, z_2^{*q},$$

and in particular

$$(3.3)\qquad u_{00}(z_1, z_2) = g_1(z_1, z_2);$$

cf. (2.9). We represent the coefficients of the system (1.4) in the form

$$(3.4)\qquad c_1(z_1, z_1^*) = \sum_{m,n=0}^{\infty} c_{mn}^{(1)}\, z_1^m\, z_1^{*n}, \qquad c_2(z_2, z_2^*) = \sum_{p,q=0}^{\infty} c_{pq}^{(2)}\, z_2^p\, z_2^{*q}.$$

Introducing the functions

$$(3.5)\qquad c_n^{(1)}(z_1) = \sum_{m=0}^{\infty} c_{mn}^{(1)}\, z_1^m, \qquad c_q^{(2)}(z_2) = \sum_{p=0}^{\infty} c_{pq}^{(2)}\, z_2^p, \qquad n = 0, 1, \ldots, \quad q = 0, 1, \ldots,$$

we may write the representation (3.4) in the form

$$(3.6)\qquad c_1(z_1, z_1^*) = \sum_{n=0}^{\infty} c_n^{(1)}(z_1)\, z_1^{*n}, \qquad c_2(z_2, z_2^*) = \sum_{q=0}^{\infty} c_q^{(2)}(z_2)\, z_2^{*q}.$$

We now insert (3.2) and (3.6) into (1.4). In order that the two resulting equations be satisfied identically in z_1^*, z_2^*, the coefficient of each product $z_1^{*n} z_2^{*q}$ in these equations must vanish. We thus obtain two systems of linear differential equations of the first order involving the functions $u_{nq}(z_1, z_2)$ and their first partial derivatives. Indeed, from the first of the equations (1.4) we find

$$(3.7\text{a})\qquad n\frac{\partial u_{nq}}{\partial z_1} + \sum_{\nu=0}^{n-1} c_{n-\nu-1}^{(1)}(z_1)\, u_{\nu q} = 0, \qquad n = 1, 2, \ldots, \quad q = 0, 1, \ldots,$$

and from the second equation (1.4)

$$(3.7\text{b})\qquad q\frac{\partial u_{nq}}{\partial z_2} + \sum_{\varkappa=0}^{q-1} c_{q-\varkappa-1}^{(2)}(z_2)\, u_{n\varkappa} = 0, \qquad n = 0, 1, \ldots, \quad q = 1, 2, \ldots.$$

Hence our problem leads to the investigation of these two systems of differential equations. We denote the equations (3.7a) and (3.7b) corresponding to a pair (n, q) by $[n, q]_a$ and $[n, q]_b$, respectively.

4. Singularities of Type S

In the subsequent sections we shall derive relations between singularities of an arbitrary function $u_{nq}(z_1, z_2)$, $n>0$, $q>0$ [*cf.* (3.1)], and those of the function $u_{00}(z_1, z_2) \equiv g_1(z_1, z_2)$ [*cf.* (2.9)]. For simplicity we shall refer to an (arbitrary) point P: (z_1, z_2) in all the statements. It is clear how this is to be understood from the standpoint of the theory of analytic functions of two complex variables. We confine ourselves to some remarks in this direction which are necessary for what follows.

Let $u_{nq}(z_1, z_2)$, $n>0$, $q>0$ [*cf.* (3.1)] be an arbitrary analytic function, defined in a simply-connected bounded domain $\mathfrak{D}$ of $z_1 z_2$-space, containing the origin. Let $P \in \mathfrak{D}$ be an arbitrary point and $\mathfrak{U}(P) < \mathfrak{D}$ a sufficiently small neighborhood of P.

Definition. *A singularity of $u_{nq}(z_1, z_2)$ at P is said to be of type S if $u_{nq}(z_1, z_2)$ can be represented in $\mathfrak{U}(P)$ in the form*

$$u_{nq}(z_1, z_2) = r_{12}(z_1, z_2) + s_2(z_2)\, r_1(z_1) + s_1(z_1)\, r_2(z_2) \tag{4.1}$$

where the functions r_{12}, r_1, and r_2 are regular at P while the functions s_1 and s_2 are singular at P.

Obviously, if $u_{nq}(z_1, z_2)$ has a singularity of type S at P then its singularities in $\mathfrak{U}(P)$ lie on portions of finitely many planes $z_1 =$ const. or $z_2 =$ const. We should stress the fact that this is an essential restriction on the type and location of the singularities. Indeed, in the case of an arbitrary analytic function defined in $\mathfrak{D}$ and singular at P the singularities of this function in $\mathfrak{U}(P)$ may lie on portions of arbitrary analytic surfaces. For example, if $f(z_1, z_2)$ is a meromorphic function in $\mathfrak{D}$ it can be represented in the (sufficiently small) neighborhood $U(P)$ in the form $f(z_1, z_2) = h_1(z_1, z_2)/h_2(z_1, z_2)$, where h_1 and h_2 are regular functions in $\mathfrak{U}(P)$ without common factor. Let $h_2 = 0$ at P. Then if $h_1 \neq 0$ at P, the function f has a pole; if $h_1 = 0$ at P, the function f has a non-essential singularity of the second kind at P; these singularities are isolated while poles lie on analytic surfaces. For further details see BEHNKE & THULLEN [*8*] and BOCHNER & MARTIN [*9*].

5. Behavior of u_{00} at Singular Points of u_{nq}. Behavior of u_{nq} at Regular Points of u_{00}

Relations between $u_{00}(z_1, z_2) \equiv g_1(z_1, z_2)$ and an arbitrary function $u_{nq}(z_1, z_2)$, $n>0$, $q>0$, may be investigated systematically as follows. We shall determine the behavior of u_{00} at singular points (Section 5) and at regular points (Section 7) of u_{nq} and, conversely, the behavior of u_{nq} at singular points (Section 6) and at regular points (Section 5) of u_{00}.

We start with the first of these four problems. The result will be that singularities of type S are the only ones of $u_{nq}(z_1, z_2)$, $n>0$, $q>0$, which may correspond to regular points of $u_{00}(z_1, z_2)$. We shall prove the following simple basic

Theorem 1. *If a function $u_{nq}(z_1, z_2)$, $n>0$, $q>0$, has a singularity at a point P which is not of type S, then $u_{00}(z_1, z_2)$ is singular at P.*

Proof. Since the coefficients $c_k(z_k, z_k^*)$, $k=1, 2$, of the system (1.4) are assumed to be entire functions, all the coefficients $c_n^{(1)}(z_1)$ and $c_q^{(2)}(z_2)$ of the differential

equations (3.7) are entire functions. Let $u_{nq}(z_1, z_2)$, $n>0$, $q>0$, be an arbitrary function which has a singularity, not of type S, at a point P. We consider the sub-systems of (3.7) consisting of the $2nq+n+q$ equations

$$[\nu, \varkappa]_a, \qquad \nu = 1, 2, \ldots, n, \qquad \varkappa = 0, 1, \ldots, q;$$

$$[\nu, \varkappa]_b, \qquad \nu = 0, 1, \ldots, n, \qquad \varkappa = 1, 2, \ldots, q.$$

At least one of the coefficients $c^{(1)}(z_1)$ and $c^{(2)}(z_2)$ of each of the two equations $[n, q]_a$ and $[n, q]_b$ cannot vanish identically, since otherwise $u_{nq}(z_1, z_2)$ would be constant. Since the singularity under consideration is not of type S, at least one of the two first partial derivatives of $u_{nq}(z_1, z_2)$ is singular at P. Hence, at least one of the functions $u_{\nu q}$, $0 \leqq \nu < n$ or $u_{n\varkappa}$, $0 \leqq \varkappa < q$, occurring in the respective equations $[n, q]_a$ and $[n, q]_b$ must be singular at P. The corresponding coefficient appears also in other equations of the sub-systems of (3.7) under consideration. By repeating the preceding conclusion we arrive at the intermediate result that at least one of the two following statements holds. (Case I) The function $u_{0q}(z_1, z_2)$ is singular at P. (Case II) The function $u_{n0}(z_1, z_2)$ is singular at P. In Case I the function $\partial u_{0q}/\partial z_2$ is also singular at P. In Case II the function $\partial u_{n0}/\partial z_1$ is also singular at P. We consider Case I. From equation $[0, q]_b$ we find that at least one of the functions $u_{0\varkappa}(z_1, z_2)$, $0 \leqq \varkappa < q$, is singular at P. Using suitable equations from among $[0, \varkappa]_b$, $0 < \varkappa < q$, we obtain the final result that $u_{00}(z_1, z_2)$ is singular at P. In Case II we have to consider the equations $[\nu, 0]_a$, $0 < \nu \leqq n$, and arrive at the same result.

Conversely, let us assume that $u_{00}(z_1, z_2)$ is regular at a point Q. Then, by integrating the equations of the above sub-systems of (3.7) we see that $u_{nq}(z_1, z_2)$ may be regular at Q or may have a singularity of type S at this point. Using (2.9) we thus have

Theorem 2. *If the associated function $g_1(z_1, z_2)$ of a solution $u(z_1, z_1^*, z_2, z_2^*)$ of* (1.4) *is regular in a simply-connected bounded domain $\mathfrak{D}$ which contains the origin, then each of the corresponding functions $u_{nq}(z_1, z_2)$, $n, q = 1, 2, \ldots$, is regular or has at most singularities of type S in $\mathfrak{D}$.*

Hence the regularity of $g_1(z_1, z_2)$ in $\mathfrak{D}$ imposes strong restrictions on the nature and distribution of possible singularities of each corresponding function $u_{nq}(z_1, z_2)$ in $\mathfrak{D}$; the only singularity surfaces which $u_{nq}(z_1, z_2)$ may have in $\mathfrak{D}$ are finitely many portions of planes $z_1 = \text{const.}$ or $z_2 = \text{const.}$ If $u_{nq}(z_1, z_2)$ is singular at one point of such a plane, it is singular at all points of this plane which lie in $\mathfrak{D}$.

It is worthwhile to mention that in the case of a solution

$$u(z, z^*) = \sum_{n=0}^{\infty} u_n(z)\, z^{*n}, \qquad u_n(z) = \sum_{m=0}^{\infty} u_{mn} z^m \tag{5.1}$$

of a single partial differential equation (1.1) the following similar but stronger result holds. If $u_0(z) \equiv g(z)$ [*cf.* (2.2)] is regular in a simply-connected bounded domain $\mathfrak{B}$ which contains the origin, then the functions $u_n(z)$, $n = 1, 2, \ldots$, are regular in $\mathfrak{B}$. This follows from (2.2) and (2.3), as was proved in [7].

A similar improvement of Theorem 2 cannot be obtained in general but only for certain trivial cases where the functions $c_k(z_k, z_k^*)$ in (1.4) are of such a special form that some of the functions $u_{nq}(z_1, z_2)$, $n>0$, $q>0$, become completely independent of $u_{00}(z_1, z_2)$.

6. Behavior of u_{nq} at Singular Points of u_{00}

If $u_{nq}(z_1, z_2)$ has a singularity of type S at a point P, then $u_{00}(z_1, z_2)$ is singular at P while some of the functions $u_{\nu\varkappa}(z_1, z_2)$, $0<\nu<n$, $0<\varkappa<q$ may be regular at P. For example, if the function $u_{22}(z_1, z_2)$ corresponding to a solution $u(z_1, z_1^*, z_2, z_2^*)$ of the system

$$u_{z_k z_k^*} + z_k^* u = 0, \qquad k = 1, 2 \tag{6.1}$$

has a singularity not of type S at a point P, then the associated function $g_1(z_1, z_2)$ of u is singular at P while $u_{11}(z_1, z_2)$ is regular at P.

Hence we have the situation that singular points of $u_{00}(z_1, z_2)$ may correspond to regular points of $u_{nq}(z_1, z_2)$, $n>0$, $q>0$.

For example, the following statement holds:

Lemma 3. *Suppose the series developments* (3.4) *of the coefficients* $c_1(z_1, z_1^*)$ *and* $c_2(z_2, z_2^*)$ *of the system* (1.4) *do not involve constant terms or terms which depend only on the respective variable* z_1 *and* z_2. *Let* z_1^{*n} *and* z_2^{*q} *be the smallest powers of* z_1^* *and* z_2^*, *respectively, occurring in* (3.4). *Then, for every solution* $u(z_1, z_1^*, z_2, z_2^*)$ *of* (1.4), *regular at the origin, the functions* $u_{\nu\varkappa}(z_1, z_2)$, $0<\nu\leq n$, $0<\varkappa\leq q$ *are entire functions.*

The proof follows from (3.7).

The fact that singular points of $u_{00}(z_1, z_2)$ may correspond to regular points of $u_{nq}(z_1, z_2)$ is of principal importance. It has the consequence that the information about the domain of regularity and other properties of a solution u of (1.4) which can be obtained from a sequence $(u_{m\,n\,p\,q})$, $n>0$, $q>0$, both fixed, is less than that which can be obtained from the sequence $(u_{m\,0\,p\,0})$. There are two different cases, as follows:

(a) If for every solution of (1.4), regular at the origin, certain functions $u_{\nu\varkappa}(z_1, z_2)$, $0<\nu\leq n$, $0<\varkappa\leq q$, are entire, these functions must be completely independent of $u_{00}(z_1, z_2) \equiv g_1(z_1, z_2)$. Obviously this holds if all the coefficients of the differential equations (3.7) relating these functions and $u_{00}(z_1, z_2)$ vanish identically. In order that *all* the functions $u_{\nu\varkappa}(z_1, z_2)$, $\nu, \varkappa = 0, 1, \ldots$ be independent of each other, the system (1.4) must reduce to the form

$$u_{z_k z_k^*} = 0, \qquad k = 1, 2. \tag{6.2}$$

(b) Certain singularities of some special associated functions $g_1(z_1, z_2)$ may coincide with zeros (of sufficiently large order) of the coefficients of (3.7). Because of this coincidence, a function $u_{nq}(z_1, z_2)$, $n>0$, $q>0$ may be regular at such a point. However, since each coefficient of (3.7) depends only on one of the two variables z_1 or z_2 this may happen only for singularities which lie on singular planes $z_1 = \text{const.}$ or $z_2 = \text{const.}$

We may sum up our result as follows:

Theorem 4. *Suppose that the function* $u_{nq}(z_1, z_2)$, $n > 0$, $q > 0$, *both fixed, and the function* $u_{00}(z_1, z_2)$ *are related by some of the equations*

$$[\nu, \varkappa]_a, \qquad \nu = 1, 2, \ldots, n, \qquad \varkappa = 0, 1, \ldots, q,$$

and

$$[\nu, \varkappa]_b, \qquad \nu = 0, 1, \ldots, n, \qquad \varkappa = 1, 2, \ldots, q;$$

cf. (3.7). *Let* $u_{00}(z_1, z_2)$ *be singular at a point* P *so that at least one of the singularity surfaces of* $u_{00}(z_1, z_2)$ *passing through* P *is not a plane* $z_1 = \text{const.}$ *or* $z_2 = \text{const.}$ *Then* $u_{nq}(z_1, z_2)$ *is singular at* P.

7. Behavior of u_{00} at Regular Points of u_{nq}

We consider an arbitrary function $u_{nq}(z_1, z_2)$, $n > 0$, $q > 0$, both fixed. Suppose this function is regular in a simply-connected bounded domain $\mathfrak{D}$ containing the origin. We ask for information about the behavior of $u_{00}(z_1, z_2)$ in $\mathfrak{D}$ which results from this assumption. If $u_{nq}(z_1, z_2)$ and $u_{00}(z_1, z_2)$ are independent (*cf.* the previous section), we cannot draw any such conclusion. In order to exclude this trivial case we assume that $u_{nq}(z_1, z_2)$ and $u_{00}(z_1, z_2)$ are related by some of the equations $[\nu, \varkappa]_a$, $\nu = 1, 2, \ldots, n$, $\varkappa = 0, 1, \ldots, q$, and $[\nu, \varkappa]_b$, $\nu = 0, 1, \ldots, n$, $\varkappa = 1, 2, \ldots, q$, *cf.* (3.7). If a function $c^{(1)}(z_1)$ or $c^{(2)}(z_2)$ does not vanish identically it always occurs in several of these equations. This fact has the following consequence. The equations which relate $u_{00}(z_1, z_2)$ and $u_{nq}(z_1, z_2)$ always contain at least one of two special classes (A) or (B), and the equations of that class relate $u_{00}(z_1, z_2)$ and $u_{nq}(z_1, z_2)$. Class (A) consists in the systems

$$(7.1) \qquad (A_1) \qquad \nu \frac{\partial u_{\nu q}}{\partial z_1} + \sum_{\lambda=0}^{\nu-1} c^{(1)}_{\nu-\lambda-1}(z_1)\, u_{\lambda q} = 0, \qquad \nu = 1, 2, \ldots, n,$$

and

$$(7.2) \qquad (A_2) \qquad \varkappa \frac{\partial u_{0\varkappa}}{\partial z_2} + \sum_{\mu=0}^{\varkappa-1} c^{(2)}_{\varkappa-\mu-1}(z_2)\, u_{0\mu} = 0, \qquad \varkappa = 1, 2, \ldots, q.$$

Class (B) consists in the systems

$$(7.3) \qquad (B_1) \qquad \nu \frac{\partial u_{\nu 0}}{\partial z_1} + \sum_{\lambda=0}^{\nu-1} c^{(1)}_{\nu-\lambda-1}(z_1)\, u_{\lambda 0} = 0, \qquad \nu = 1, 2, \ldots, n,$$

and

$$(7.4) \qquad (B_2) \qquad \varkappa \frac{\partial u_{n\varkappa}}{\partial z_2} + \sum_{\mu=0}^{\varkappa-1} c^{(2)}_{\varkappa-\mu-1}(z_2)\, u_{n\mu} = 0, \qquad \varkappa = 1, 2, \ldots, q.$$

Whether class (A) or class (B) relates the functions $u_{00}(z_1, z_2)$ and $u_{nq}(z_1, z_2)$ under consideration depends on the form of the coefficients $c_k(z_k, z_k^*)$ of (1.4). We may consider class (A) only, since for class (B) the manner of reasoning is exactly the same.

We eliminate from the system (A_1) the functions $u_{\nu q}$, $\nu = 1, 2, \ldots, n-1$, and their derivatives. For this purpose we consider (A_1) as a system of n homo-

geneous equations in the $2n-1$ unknowns

$$u_{1q}, \frac{\partial u_{1q}}{\partial z_1}, u_{2q}, \frac{\partial u_{2q}}{\partial z_1}, \ldots, u_{n-1,q}, \frac{\partial u_{n-1,q}}{\partial z_1}, \text{ and } 1,$$

where the coefficient of the "unknown" 1 involves all the terms containing the functions u_{0q}, u_{nq} and their partial derivatives with respect to z_1. If we differentiate each equation of (A_1) $(n-1)$ times with respect to z_1, we obtain a system $(\tilde{A}_1)$ which consists in n^2 equations in the same number of unknowns; the determinant of this system will be denoted by $D(\tilde{A}_1)$. If $D(\tilde{A}_1) \equiv 0$, the further procedure of elimination has to be applied to a suitable sub-system of (A_1) for which the determinant of the corresponding enlarged system (obtained by suitable differentiation) does not vanish identically. It is not necessary to pay further attention to the exceptional case $D(\tilde{A}_1) \equiv 0$, and we may assume that $D(\tilde{A}_1) \not\equiv 0$. The equation

$$D(\tilde{A}_1) = 0 \tag{7.5}$$

is a linear differential equation involving only the functions $u_{0q}(z_1, z_2)$ and $u_{nq}(z_1, z_2)$ and their partial derivatives with respect to z_1. We shall write this equation explicitly. Let $[1, 1]$, $[2, 1]$, ..., $[n, 1]$ denote the equations of (A_1) and $[\nu, \varrho]$ the equation obtained by differentiating equation $[\nu, 1]$ $\varrho-1$ times. We may write the enlarged system $(\tilde{A}_1)$ in the form

$$[\nu, \varrho]: \quad \nu \frac{\partial^\varrho u_{\nu q}}{\partial z_1^\varrho} + \sum_{\alpha=1}^{\nu-1} \sum_{\beta=1}^{\varrho-1} \binom{\varrho-1}{\beta} \frac{d^{\varrho-\beta-1} c_{\nu-\alpha-1}^{(1)}}{d z_1^{\varrho-\beta-1}} \frac{\partial^\beta u_{\alpha q}}{\partial z_1^\beta} = 0, \tag{7.6}$$
$$\nu, \varrho = 1, 2, \ldots, n.$$

We arrange each of these n^2 equations so that the terms involving the functions u_{0q} and u_{nq} and their derivatives occur first. If we develop the determinant $D(\tilde{A}_1)$ by its first column then (7.5) takes the form

$$\sum_{s=0}^{n} \left(a_s(z_1) \frac{\partial^s u_{0q}}{\partial z_1^s} + b_s(z_1) \frac{\partial^s u_{nq}}{\partial z_1^s} \right) = 0; \tag{7.7}$$

in this representation,

$$a_s = \begin{cases} \sum_{\nu,\varrho=1}^{n} D_{\nu\varrho} p_{\nu\varrho s} & (0 \leqq s < n), \\ 0 & (s = n), \end{cases} \qquad b_s = \begin{cases} 0 & (s = 0), \\ n D_{ns} & (0 < s \leqq n); \end{cases}$$

$D_{\sigma\tau}$ is the cofactor in $D(\tilde{A}_1)$ of the element in the first column and in the row corresponding to equation $[\sigma, \tau]$, and

$$p_{\nu\varrho s} = \begin{cases} 0 & (s \geqq \varrho), \\ \binom{\varrho-1}{s} \frac{d^{\varrho-s-1} c_{\nu-1}^{(1)}}{d z_1^{\varrho-s-1}} & (0 \leqq s < \varrho) \end{cases}$$

is the coefficient of the function $\partial^s u_{0q}/\partial z_1^s$ in the equation $[\nu, \varrho]$.

For example, in the case of the functions $u_{2q}(z_1, z_2)$, (7.7) has the form

$$\left(c_0^{(1)^2} - c_0^{(1)} \frac{dc_1^{(1)}}{dz_1} + \frac{dc_0^{(1)}}{dz_1} c_1^{(1)}\right) u_{0q} - c_0^{(1)} c_1^{(1)} \frac{\partial u_{0q}}{\partial z_1} + 2 \frac{dc_0^{(1)}}{dz_1} \frac{\partial u_{2q}}{\partial z_1} - 2c_0^{(1)} \frac{\partial^2 u_{2q}}{\partial z_1^2} = 0.$$

We may derive some general properties of the coefficients of (7.7) as follows. If all the coefficients b_s vanish identically, the same holds for the coefficients a_s, since otherwise u_{0q} would be the same function for solutions u of (1.4) corresponding to any function u_{nq} whatsoever. That is, $D(\tilde{A}_1) \equiv 0$, but this case was excluded. If the functions a_s vanish identically but at least one function b_s does not vanish identically, then (7.7) is a differential equation in u_{nq} which does not involve u_{0q}. That is, u_{nq} and u_{0q} are independent; this leads to a contradiction to the above assumptions. Hence, $a_s \not\equiv 0$ for at least one value of $s = 0, 1, \ldots, n-1$, and also $b_s \not\equiv 0$ for at least one value of $s = 1, 2, \ldots, n$. If $s = m$ $(< n)$ is the largest value for which $a_s \not\equiv 0$, then (7.7), considered as a differential equation in u_{0q}, has the order m.

In a similar manner we can transform the system (A_2) into a single linear differential equation

$$\sum_{s=0}^{q} \left[w_s(z_2) \frac{\partial^s u_{00}}{\partial z_2^s} + h_s(z_2) \frac{\partial^s u_{0q}}{\partial z_2^s} \right] = 0. \tag{7.8}$$

The preceding general properties hold also for the coefficients of this equation.

Now, the solution $w(z)$ of an ordinary linear non-homogeneous differential equation $L(w) = f(z)$ with analytic coefficients and $f(z)$ analytic is regular in the intersection of the domains of regularity of these coefficients and $f(z)$. Since the coefficients of (7.7) and (7.8) are entire functions of the respective variable we obtain

Theorem 5. *Suppose the coefficients $c_k(z_k, z_k^*)$ of (1.4) are such that, for a solution u, the corresponding function $u_{nq}(z_1, z_2)$, $n > 0$, $q > 0$, both arbitrary and fixed, is related to the corresponding function $u_{00}(z_1, z_2)$ by some of the equations $[\nu, \varkappa]_a$, $\nu = 1, 2, \ldots, n$, $\varkappa = 0, 1, \ldots, q$ and $[\nu, \varkappa]_b$, $\nu = 0, 1, \ldots, n$, $\varkappa = 1, 2, \ldots, q$* [cf. (3.7)]. *Then the following statement holds. If $u_{nq}(z_1, z_2)$ is regular in a simply-connected bounded domain $\mathfrak{D}$ containing the origin, then $u_{00}(z_1, z_2)$ is regular in a sub-domain $\mathfrak{D}'$ of $\mathfrak{D}$ which differs from $\mathfrak{D}$ by at most finitely many portions of the planes $z_1 =$ const. and $z_2 =$ const.*

This means that if u_{nq} and u_{00} are not independent and u_{nq} is regular in a domain $\mathfrak{D}$ of the above type, the function u_{00} may have some singularities in $\mathfrak{D}$, but only singularities of a very simple type. These singularities, considered in the intersection of $\mathfrak{D}$ and a plane $z_1 =$ const. or $z_2 =$ const., are either isolated or each point of $\mathfrak{D}$ in this plane is a singular point of u_{00}.

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